

Combined large- N_c and heavy-quark operator analysis for the chiral Lagrangian with charmed baryons

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Abstract

The chiral $SU(3)$ Lagrangian with charmed baryons of spin $J^P = 1/2^+$ and $J^P = 3/2^+$ is analyzed. We consider all counter terms that are relevant at next-to-next-to-next-to-leading order (N^3LO) in a chiral extrapolation of the charmed baryon masses. At N^2LO we find 16 low-energy parameters. There are 3 mass parameters for the anti-triplet and the two sextet baryons, 6 parameters describing the meson-baryon vertices and 7 symmetry breaking parameters. The heavy-quark spin symmetry predicts four sum rules for the meson-baryon vertices and degenerate masses for the two baryon sextet fields. Here a large- N_c operator analysis at NLO suggests the relevance of one further spin-symmetry breaking parameter. Going from N^2LO to N^3LO adds 17 chiral symmetry breaking parameters and 24 symmetry preserving parameters. For the leading symmetry conserving two-body counter terms involving two baryon fields and two Goldstone boson fields we find 36 terms. While the heavy-quark spin symmetry leads to $36 - 16 = 20$ sum rules, an expansion in $1/N_c$ at next-to-leading order (NLO) generates $36 - 7 = 29$ parameter relations. A combined expansion leaves 3 unknown parameters only. For the symmetry breaking counter terms we find 17 terms, for which there are $17 - 9 = 8$ sum rules from the heavy-quark spin symmetry and $17 - 5 = 12$ sum rules from a $1/N_c$ expansion at NLO.

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I. INTRODUCTION

The chiral $SU(3)$ Lagrangian with the charmed baryons has been used to study baryon resonances with $J^P = \frac{1}{2}^-$ and $\frac{3}{2}^-$ quantum numbers [1–6]. So far such studies rely on the leading order interaction characterized by the Weinberg-Tomozawa theorem [1] or they are based on phenomenological assumptions like meson-exchange dynamics [2–6]. Such systems are of considerable complexity due to the possibility of charm exchange reactions like for instance $\pi\Sigma_c \rightarrow \bar{D}\Sigma$. As been observed already in [2, 3] the generated resonance spectrum decouples approximately into two sectors. The first sector is dominated by the coupled-channel interaction of the D mesons with charm zero baryons and the second one by the coupled-channel interaction of the Goldstone bosons with charmed baryons. Though there are already some partial results available in the literature it is an open challenge to derive quantitatively the spectrum of open-charm baryons in terms of realistic interactions derived from the chiral Lagrangian.

While the leading order interaction of the Goldstone bosons with any hadron is dictated by the chiral symmetry, this is not true for the interaction of the D mesons with baryons. The D mesons interact with baryons via local counter terms that are unconstrained by chiral symmetry. Only the long-range part of the interaction is controlled by chiral interactions, the short range part needs to be parameterized in terms of a priori unknown contact terms. In a previous work [7] such counter terms were studied systematically in the light of the heavy-quark symmetry and large- N_c sum rules. The 26 counter terms were correlated so that only 7 unknown parameters remain. A corresponding analysis for the counter terms describing the residual short-range interaction of the Goldstone bosons with the charmed baryons has not been performed. Such an analysis is relevant also for an accurate flavour $SU(3)$ chiral extrapolation of the charmed ground-state baryon masses as they are currently evaluated by various lattice QCD groups. First partial results are found in [8–13].

Recently it was demonstrated that a systematic and accurate flavour $SU(3)$ chiral extrapolation of the baryon octet and decuplet states with zero charm content is feasible [14–17]. Based on the chiral Lagrangian formulated with spin 1/2 and 3/2 fields the available lattice data on the baryon masses was reproduced and accurate predictions for the size of the low-energy parameters relevant at $N^3\text{LO}$ were made. Such an analysis was made possible only by the availability of sum rules for the low-energy parameters as they arise in the limit of

a large number colors (N_c) in QCD [18]. The latter provided a large parameter reduction that allowed fits at N³LO to the lattice data set that are significant.

The purpose of the present work is to pave the way towards a corresponding analysis of the upcoming lattice data for the charmed baryons. In order to derive sum rules for the symmetry conserving low-energy constants we study matrix elements of current-current correlation functions in the charmed baryon states [7, 18]. The technology developed in [19–21] will be applied. The implications of heavy-quark symmetry on the counter terms can be worked out using a suitable multiplet representation of the charmed baryons [22–24].

The paper is organized as follows. In section II the chiral Lagrangian will be constructed. It follows a section where the consequences of the heavy-quark spin symmetry on the low-energy constants of the chiral Lagrangian are worked out. In section IV a corresponding large- N_c operators analysis is presented. In section V a summary of the main results is given.

II. CHIRAL LAGRANGIAN WITH CHARMED BARYON FIELDS

The construction rules for the chiral $SU(3)$ Lagrangian density are recalled. For more technical details see for example [22–25]. The basic building blocks of the chiral Lagrangian are

$$U_\mu = \frac{1}{2} e^{-i\frac{\Phi}{2f}} \left(\partial_\mu e^{i\frac{\Phi}{f}} \right) e^{-i\frac{\Phi}{2f}} - \frac{i}{2} e^{-i\frac{\Phi}{2f}} r_\mu e^{+i\frac{\Phi}{2f}} + \frac{i}{2} e^{+i\frac{\Phi}{2f}} l_\mu e^{-i\frac{\Phi}{2f}},$$

$$\chi_\pm = \frac{1}{2} \left(e^{+i\frac{\Phi}{2f}} \chi_0 e^{+i\frac{\Phi}{2f}} \pm e^{-i\frac{\Phi}{2f}} \chi_0 e^{-i\frac{\Phi}{2f}} \right), \quad B_{[\bar{3}]}, \quad B_{[6]}, \quad B_{[6]}^\mu, \quad (1)$$

where we include the pseudoscalar meson octet fields $\Phi(J^P=0^-)$, the baryon fields $B_{[\bar{3}]}(J^P=\frac{1}{2}^+)$, $B_{[6]}(J^P=\frac{1}{2}^+)$ fields $B_{[6]}^\mu(J^P=\frac{3}{2}^+)$. Explicit chiral symmetry-breaking is included in terms of scalar and pseudoscalar fields χ_\pm . They introduce the scalar and pseudo-scalar classical source functions s and p with

$$\chi_0 = 2 B_0 \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} + 2 B_0 (s + i p), \quad (2)$$

and the quark mass matrix of QCD [26, 27]. The classical source functions r_μ and l_μ in (1) are linear combinations of the vector and axial-vector sources with $r_\mu = v_\mu + a_\mu$ and $l_\mu = v_\mu - a_\mu$.

It is convenient to decompose the fields into their isospin multiplets. The fields can be written in terms of isospin multiplet fields

$$\begin{aligned}
\Phi &= \tau \cdot \pi(140) + \alpha^\dagger \cdot K(494) + K^\dagger(494) \cdot \alpha + \eta(547) \lambda_8, \\
\sqrt{2} B_{[\bar{3}]} &= \frac{1}{\sqrt{2}} \alpha^\dagger \cdot \Xi_c(2470) - \frac{1}{\sqrt{2}} \Xi_c^t(2470) \cdot \alpha + i \tau_2 \Lambda_c(2284), \\
\sqrt{2} B_{[6]} &= \frac{1}{\sqrt{2}} \alpha^\dagger \cdot \Xi_c(2580) + \frac{1}{\sqrt{2}} \Xi_c^t(2580) \cdot \alpha + \Sigma_c(2453) \cdot \tau i \tau_2 \\
&\quad + \frac{\sqrt{2}}{3} (1 - \sqrt{3} \lambda_8) \Omega_c(2704), \\
\sqrt{2} B_{[6]}^\mu &= \frac{1}{\sqrt{2}} \alpha^\dagger \cdot \Xi_c^\mu(2646) + \frac{1}{\sqrt{2}} \Xi_c^{t,\mu}(2646) \cdot \alpha + \Sigma_c^\mu(2518) \cdot \tau i \tau_2 \\
&\quad + \frac{\sqrt{2}}{3} (1 - \sqrt{3} \lambda_8) \Omega_c^\mu(2770), \\
\alpha^\dagger &= \frac{1}{\sqrt{2}} (\lambda_4 + i \lambda_5, \lambda_6 + i \lambda_7), \quad \tau = (\lambda_1, \lambda_2, \lambda_3),
\end{aligned} \tag{3}$$

where the matrices λ_i are the standard Gell-Mann generators of the $SU(3)$ algebra. The numbers in the brackets recall the approximate masses of the particles in units of MeV.

The chiral Lagrangian consists of all possible interaction terms, formed with the fields U^μ , $B_{[\bar{3}]}$, $B_{[6]}$, $B_{[6]}^\mu$ and χ_\pm . Derivatives of the fields must be included in compliance with the local chiral $SU(3)$ symmetry. This leads to the notion of a covariant derivative D_μ . For the flavour octet field U_ν and the flavour symmetric sextet and flavour antisymmetric anti-triplet fields $B_{[6]}$, $B_{[6]}^\nu$ and $B_{[\bar{3}]}$ the covariant derivative D_μ acts as follows

$$\begin{aligned}
(D_\mu U_\nu)_b^a &= \partial_\mu U_{\nu,b}^a + \Gamma_{\mu,l}^a U_{\nu,b}^l - \Gamma_{\mu,b}^l U_{\nu,l}^a, \\
(D_\mu B_{[6]})^{ab} &= \partial_\mu B^{ab} + \Gamma_{\mu,l}^a B_{[6]}^{lb} + \Gamma_{\mu,l}^b B_{[6]}^{al}, \\
(D_\mu B_{[\bar{3}]})^{ab} &= \partial_\mu B_{[\bar{3}]}^{ab} + \Gamma_{\mu,l}^a B_{[\bar{3}]}^{lb} + \Gamma_{\mu,l}^b B_{[\bar{3}]}^{al},
\end{aligned} \tag{4}$$

with the chiral connection $\Gamma_\mu = -\Gamma_\mu^\dagger$ given by

$$\Gamma_\mu = \frac{1}{2} e^{-i \frac{\Phi}{2f}} \left[\partial_\mu - i (v_\mu + a_\mu) \right] e^{+i \frac{\Phi}{2f}} + \frac{1}{2} e^{+i \frac{\Phi}{2f}} \left[\partial_\mu - i (v_\mu - a_\mu) \right] e^{-i \frac{\Phi}{2f}}.$$

The chiral Lagrangian is a powerful tool, once it is combined with appropriate counting rules leading to a systematic approximation strategy. In the following we construct the

leading order (LO) terms involving the three baryon fields. There are the kinetic terms

$$\begin{aligned}
\mathcal{L}^{(1)} = & \text{tr } \bar{B}_{[6]} (\gamma^\mu i D_\mu - M_{[6]}^{1/2}) B_{[6]} - \text{tr } \left(\bar{B}_{[6]}^\mu ([i \not{D} - M_{[6]}^{3/2}] g_{\mu\nu} - i (\gamma_\mu D_\nu + \gamma_\nu D_\mu) \right. \\
& \left. + \gamma_\mu [i \not{D} + M_{[6]}^{3/2}] \gamma_\nu) B_{[6]}^\nu \right) + \text{tr } \bar{B}_{[\bar{3}]} (\gamma^\mu i D_\mu - M_{[\bar{3}]}^{1/2}) B_{[\bar{3}]} \\
& + F_{[66]} \text{tr } \bar{B}_{[6]} \gamma^\mu \gamma_5 i U_\mu B_{[6]} + F_{[\bar{3}\bar{3}]} \text{tr } \bar{B}_{[\bar{3}]} \gamma^\mu \gamma_5 i U_\mu B_{[\bar{3}]} \\
& + F_{[\bar{3}6]} \text{tr } \left(\bar{B}_{[6]} \gamma^\mu \gamma_5 i U_\mu B_{[\bar{3}]} + \text{h.c.} \right) \\
& + C_{[66]} \text{tr } \left(\bar{B}_{[6]}^\mu i U_\mu B_{[6]} + \text{h.c.} \right) + C_{[\bar{3}6]} \text{tr } \left(\bar{B}_{[6]}^\mu i U_\mu B_{[\bar{3}]} + \text{h.c.} \right) \\
& - H_{[66]} \text{tr } \bar{B}_{[6]}^\alpha g_{\alpha\beta} \gamma^\mu \gamma_5 i U_\mu B_{[6]}^\beta, \tag{5}
\end{aligned}$$

and 6 structures which parameterizes the three-point interactions of the Goldstone bosons with the charmed baryon fields [22, 23]. From the kinetic terms one can read off the two-body Weinberg-Tomozawa interaction terms on which the coupled-channel computation of [1] rests. It follows upon an expansion in powers of the Goldstone boson fields. At leading order in a chiral expansion, the bare masses $M_{[6]}^{1/2}$, $M_{[6]}^{3/2}$ and $M_{[\bar{3}]}^{1/2}$ may be identified with the flavor average of the sextet and antitriplet baryon masses.

The main goal of this work is to derive correlations amongst the low-energy parameters introduced in the chiral Lagrangian as they follow from a $1/N_c$ expansion. For that purpose we consider the axial-vector and scalar currents,

$$A_\mu^{(a)}(x) = \bar{\Psi}(x) \gamma_\mu \gamma_5 \frac{\lambda_a}{2} \Psi(x), \quad S^{(a)}(x) = \bar{\Psi}(x) \frac{\lambda_a}{2} \Psi(x), \tag{6}$$

in baryon matrix elements, where we recall their definitions in terms of the Heisenberg quark-field operators $\Psi(x)$. With λ_a we denote the Gell-Mann matrices supplemented with a singlet matrix $\lambda_0 = \sqrt{2/3} \mathbf{1}$. Such matrix elements can be analyzed systematically in the $1/N_c$ expansion [20, 21]. Given the chiral Lagrangian, it is well defined how to derive the contribution of a given term to such matrix elements. The classical matrices of source functions, $a_\mu(x)$ and $s(x)$, enters the chiral Lagrangian via the building block

$$U_\mu = \frac{i}{2f} \partial_\mu \Phi - i a_\mu + \dots, \quad \chi_+ = 2 B_0 s + \dots, \tag{7}$$

where for notational simplicity in the following we put $B_0 = 1/2$.

For our matching purposes it suffices to take matrix elements in the flavour $SU(3)$ limit where the physical baryon states

$$|p, ij_\pm, S, \chi\rangle, \tag{8}$$

are specified by the momentum p and the flavor indices $i, j, k = 1, 2, 3$. The spin S and the spin-polarization are $\chi = 1, 2$ for the spin one-half ($S = 1/2$) and $\chi = 1, \dots, 4$ for the spin three-half states ($S = 3/2$). The flavour sextet and the anti-triplet are discriminated by their symmetric (index $+$) and anti-symmetric (index $-$) behaviour under the exchange of $i \leftrightarrow j$.

From the leading order chiral Lagrangian (5) the matrix elements of the axial-vector current are readily computed at tree-level

$$\begin{aligned}
\langle \bar{p}, mn_+, \tfrac{1}{2}, \bar{\chi} | A_i^{(a)}(0) | p, kl_+, \tfrac{1}{2}, \chi \rangle &= \sigma_{\bar{\chi}\chi}^{(i)} F_{[66]} \Lambda_{(kl)_+}^{(a), (mn)_+} + \dots, \\
\langle \bar{p}, mn_-, \tfrac{1}{2}, \bar{\chi} | A_i^{(a)}(0) | p, kl_-, \tfrac{1}{2}, \chi \rangle &= \sigma_{\bar{\chi}\chi}^{(i)} F_{[\bar{3}3]} \Lambda_{(kl)_-}^{(a), (mn)_-} + \dots, \\
\langle \bar{p}, mn_-, \tfrac{1}{2}, \bar{\chi} | A_i^{(a)}(0) | p, kl_+, \tfrac{1}{2}, \chi \rangle &= \sigma_{\bar{\chi}\chi}^{(i)} F_{[\bar{3}6]} \Lambda_{(kl)_+}^{(a), (mn)_-} + \dots, \\
\langle \bar{p}, mn_+, \tfrac{3}{2}, \bar{\chi} | A_i^{(a)}(0) | p, kl_+, \tfrac{1}{2}, \chi \rangle &= -S_{\bar{\chi}\chi}^{(i)} C_{[66]} \Lambda_{(kl)_+}^{(a), (mn)_+} + \dots, \\
\langle \bar{p}, mn_+, \tfrac{3}{2}, \bar{\chi} | A_i^{(a)}(0) | p, kl_-, \tfrac{1}{2}, \chi \rangle &= -S_{\bar{\chi}\chi}^{(i)} C_{[\bar{3}6]} \Lambda_{(kl)_-}^{(a), (mn)_+} + \dots, \\
\langle \bar{p}, mn_+, \tfrac{3}{2}, \bar{\chi} | A_i^{(a)}(0) | p, kl_+, \tfrac{3}{2}, \chi \rangle &= (\vec{S} \sigma^{(i)} \vec{S}^\dagger)_{\bar{\chi}\chi} H_{[66]} \Lambda_{(kl)_+}^{(a), (mn)_+} + \dots, \quad (9)
\end{aligned}$$

where we introduced convenient flavour and spin structures

$$\begin{aligned}
\Lambda_{(ij)_\pm}^{(a), (mn)_\pm} &= \frac{1}{4} \left(\lambda_{mi}^{(a)} \delta_{nj} \pm \lambda_{ni}^{(a)} \delta_{mj} \pm \lambda_{mj}^{(a)} \delta_{ni} + \lambda_{nj}^{(a)} \delta_{mi} \right), \\
\Lambda_{(ij)_\pm}^{(a), (mn)_\mp} &= \frac{1}{4} \left(\lambda_{mi}^{(a)} \delta_{nj} \mp \lambda_{ni}^{(a)} \delta_{mj} \pm \lambda_{mj}^{(a)} \delta_{ni} - \lambda_{nj}^{(a)} \delta_{mi} \right), \\
S_i^\dagger S_j &= \delta_{ij} - \frac{1}{3} \sigma_i \sigma_j, \quad S_i \sigma_j - S_j \sigma_i = -i \varepsilon_{ijk} S_k, \quad \vec{S} \cdot \vec{S}^\dagger = \mathbf{1}_{(4 \times 4)}, \\
\vec{S}^\dagger \cdot \vec{S} &= 2 \mathbf{1}_{(2 \times 2)}, \quad \vec{S} \cdot \vec{\sigma} = 0, \quad \varepsilon_{ijk} S_i S_j^\dagger = i \vec{S} \sigma_k \vec{S}^\dagger. \quad (10)
\end{aligned}$$

The dots in (12) represent additional terms that are suppressed as the three momenta \bar{p} and p approach zero. Such correction terms are not relevant for our matching purposes. Here we also assumed degenerate baryon masses for the baryon states as they arise in the large- N_c limit. The well known result (10) settles our conventions and notations and is also conveniently matched to the results of a large- N_c operator analysis. For the latter it suffices to study the spatial components of the axial-vector current only.

At next-to-leading order (NLO) there are symmetry conserving and symmetry breaking

terms. We identify 7 symmetry breaking counter terms

$$\begin{aligned}
\mathcal{L}_\chi^{(2)} = & b_{1,[\bar{3}\bar{3}]} \text{tr} (\bar{B}_{[\bar{3}]} B_{[\bar{3}]}) \text{tr} (\chi_+) + b_{2,[\bar{3}\bar{3}]} \text{tr} (\bar{B}_{[\bar{3}]} \chi_+ B_{[\bar{3}]}) + b_{1,[\bar{3}6]} \text{tr} (\bar{B}_{[6]} \chi_+ B_{[\bar{3}]} + \text{h.c.}) \\
& + b_{1,[66]} \text{tr} (\bar{B}_{[6]} B_{[6]}) \text{tr} (\chi_+) + b_{2,[66]} \text{tr} (\bar{B}_{[6]} \chi_+ B_{[6]}) \\
& - d_{1,[66]} \text{tr} (g_{\mu\nu} \bar{B}_{[6]}^\mu B_{[6]}^\nu) \text{tr} (\chi_+) - d_{2,[66]} \text{tr} (g_{\mu\nu} \bar{B}_{[6]}^\mu \chi_+ B_{[6]}^\nu) .
\end{aligned} \tag{11}$$

From the leading order chiral Lagrangian (11) the matrix elements of the scalar current are readily computed at tree-level. Here we consider the singlet ($a = 0$) and octet ($a = 1, \dots, 8$) components with

$$\begin{aligned}
\langle \bar{p}, mn_+, \frac{1}{2}, \bar{\chi} | S^{(a)}(0) | p, kl_+, \frac{1}{2}, \chi \rangle &= \delta_{a0} \sqrt{\frac{3}{2}} b_{1,[66]} \delta_{(kl)_+}^{(mn)_+} + b_{2,[66]} \Lambda_{(kl)_+}^{(a), (mn)_+} + \dots , \\
\langle \bar{p}, mn_-, \frac{1}{2}, \bar{\chi} | S^{(a)}(0) | p, kl_-, \frac{1}{2}, \chi \rangle &= \delta_{a0} \sqrt{\frac{3}{2}} b_{1,[\bar{3}\bar{3}]} \delta_{(kl)_-}^{(mn)_-} + b_{2,[\bar{3}\bar{3}]} \Lambda_{(kl)_-}^{(a), (mn)_-} + \dots , \\
\langle \bar{p}, mn_-, \frac{1}{2}, \bar{\chi} | S^{(a)}(0) | p, kl_+, \frac{1}{2}, \chi \rangle &= b_{1,[\bar{3}6]} \Lambda_{(kl)_+}^{(a), (mn)_-} + \dots \\
\langle \bar{p}, mn_+, \frac{3}{2}, \bar{\chi} | S^{(a)}(0) | p, kl_+, \frac{1}{2}, \chi \rangle &= 0 + \dots , \\
\langle \bar{p}, mn_+, \frac{3}{2}, \bar{\chi} | S^{(a)}(0) | p, kl_-, \frac{1}{2}, \chi \rangle &= 0 + \dots , \\
\langle \bar{p}, mn_+, \frac{3}{2}, \bar{\chi} | S^{(a)}(0) | p, kl_+, \frac{3}{2}, \chi \rangle &= \delta_{a0} \sqrt{\frac{3}{2}} d_{1,[66]} \delta_{(kl)_+}^{(mn)_+} + d_{2,[66]} \Lambda_{(kl)_-}^{(a), (mn)_+} + \dots ,
\end{aligned} \tag{12}$$

where we use the notations introduced in (10) with

$$\lambda_0 = \sqrt{\frac{2}{3}} \mathbf{1} , \quad \delta_{(ij)_\pm}^{(mn)_\pm} = \frac{1}{2} \left(\delta_{mi} \delta_{nj} \pm \delta_{ni} \delta_{mj} \right) . \tag{13}$$

Like in (9) the dots in (12) represent additional terms that are not relevant in this work.

There are 36 symmetry preserving terms counter terms in $\mathcal{L}^{(2)}$. Following [18, 28] the symmetry conserving counter term are classified according to their Dirac structure.

$$\mathcal{L}^{(2)} = \mathcal{L}_\chi^{(2)} + \mathcal{L}^{(S)} + \mathcal{L}^{(V)} + \mathcal{L}^{(A)} + \mathcal{L}^{(T)} . \tag{14}$$

A complete list relevant at second order is

$$\begin{aligned}
\mathcal{L}^{(S)} = & g_{0,[\bar{3}\bar{3}]}^{(S)} \text{tr} (\bar{B}_{[\bar{3}]} B_{[\bar{3}]}) \text{tr} (U_\mu U^\mu) + g_{D,[\bar{3}\bar{3}]}^{(S)} \text{tr} (\bar{B}_{[\bar{3}]} \{U_\mu, U^\mu\} B_{[\bar{3}]}) \\
& + g_{0,[66]}^{(S)} \text{tr} (\bar{B}_{[6]} B_{[6]}) \text{tr} (U_\mu U^\mu) + g_{1,[66]}^{(S)} \text{tr} (\bar{B}_{[6]} U^\mu B_{[6]} U_\mu^T) \\
& + g_{D,[66]}^{(S)} \text{tr} (\bar{B}_{[6]} \{U_\mu, U^\mu\} B_{[6]}) + g_{1,[\bar{3}6]}^{(S)} \text{tr} (\bar{B}_{[6]} U^\mu B_{[\bar{3}]} U_\mu^T + \text{h.c.}) \\
& + g_{D,[\bar{3}6]}^{(S)} \text{tr} (\bar{B}_{[6]} \{U_\mu, U^\mu\} B_{[\bar{3}]} + \text{h.c.}) \\
& + h_{0,[66]}^{(S)} \text{tr} (\bar{B}_{[6]}^\mu g_{\mu\nu} B_{[6]}^\nu) \text{tr} (U_\alpha U^\alpha) + h_{1,[66]}^{(S)} \text{tr} (\bar{B}_{[6]}^\mu B_{[6]}^\nu) \text{tr} (U_\mu U_\nu) \\
& + h_{2,[66]}^{(S)} \text{tr} (\bar{B}_{[6]}^\mu g_{\mu\nu} \{U^\alpha, U_\alpha\} B_{[6]}^\nu) + h_{3,[66]}^{(S)} \text{tr} (\bar{B}_{[6]}^\mu \{U_\mu, U_\nu\} B_{[6]}^\nu)
\end{aligned}$$

$$\begin{aligned}
& + h_{4,[66]}^{(S)} \text{tr} (\bar{B}_{[6]}^\mu g_{\mu\nu} U^\alpha B_{[6]}^\nu U_\alpha^T) + h_{5,[66]}^{(S)} \text{tr} (\bar{B}_{[6]}^\mu U_\nu B_{[6]}^\nu U_\mu^T + \bar{B}_{[6]}^\mu U_\mu B_{[6]}^\nu U_\nu^T), \\
\mathcal{L}^{(V)} = & g_{0,[33]}^{(V)} \text{tr} (\bar{B}_{[3]} i \gamma^\alpha (D^\beta B_{[3]}) \text{tr} (U_\beta U_\alpha) + \text{h.c.}) \\
& + g_{1,[33]}^{(V)} \text{tr} (\bar{B}_{[3]} i \gamma^\alpha U_\beta (D^\beta B_{[3]}) U_\alpha^T + \bar{B}_{[3]} i \gamma^\alpha U_\alpha (D^\beta B_{[3]}) U_\beta^T + \text{h.c.}) \\
& + g_{D,[33]}^{(V)} \text{tr} (\bar{B}_{[3]} i \gamma^\alpha \{U_\alpha, U_\beta\} (D^\beta B_{[3]}) + \text{h.c.}) \\
& + g_{1,[36]}^{(V)} \text{tr} (\bar{B}_{[6]} i \gamma^\alpha U_\alpha (D^\beta B_{[3]}) U_\beta^T + \bar{B}_{[6]} i \gamma^\alpha U_\beta (D^\beta B_{[3]}) U_\alpha^T + \text{h.c.}) \\
& + g_{D,[36]}^{(V)} \text{tr} (\bar{B}_{[6]} i \gamma^\alpha \{U_\alpha, U_\beta\} (D^\beta B_{[3]}) - (D^\beta \bar{B}_{[6]}) i \gamma^\alpha \{U_\alpha, U_\beta\} B_{[3]} + \text{h.c.}) \\
& + g_{0,[66]}^{(V)} (\text{tr} (\bar{B}_{[6]} i \gamma^\alpha (D^\beta B_{[6]})) \text{tr} (U_\beta U_\alpha) + \text{h.c.}) \\
& + g_{1,[66]}^{(V)} \text{tr} (\bar{B}_{[6]} i \gamma^\alpha U_\beta (D^\beta B_{[6]}) U_\alpha^T + \bar{B}_{[6]} i \gamma^\alpha U_\alpha (D^\beta B_{[6]}) U_\beta^T + \text{h.c.}) \\
& + g_{D,[66]}^{(V)} \text{tr} (\bar{B}_{[6]} i \gamma^\alpha \{U_\alpha, U_\beta\} (D^\beta B_{[6]}) + \text{h.c.}) \\
& + h_{0,[66]}^{(V)} \text{tr} (\bar{B}_{[6]}^\mu g_{\mu\nu} i \gamma^\alpha (D^\beta B_{[6]}^\nu) \text{tr} (U_\alpha U_\beta) + \text{h.c.}) \\
& + h_{1,[66]}^{(V)} \text{tr} (\bar{B}_{[6]}^\mu g_{\mu\nu} i \gamma^\alpha U_\beta (D^\beta B_{[6]}^\nu) U_\alpha^T + \bar{B}_{[6]}^\mu g_{\mu\nu} i \gamma^\alpha U_\alpha (D^\beta B_{[6]}^\nu) U_\beta^T + \text{h.c.}) \\
& + h_{2,[66]}^{(V)} \text{tr} (\bar{B}_{[6]}^\mu g_{\mu\nu} i \gamma^\alpha \{U_\alpha, U_\beta\} (D^\beta B_{[6]}^\nu) + \text{h.c.}), \\
\mathcal{L}^{(A)} = & f_{0,[66]}^{(A)} \text{tr} (\bar{B}_{[6]}^\mu \gamma^\nu \gamma_5 B_{[6]} \text{tr} (U_\nu U_\mu) + \text{h.c.}) \\
& + f_{1,[66]}^{(A)} \text{tr} (\bar{B}_{[6]}^\mu \gamma^\nu \gamma_5 U_\nu B_{[6]} U_\mu^T + \bar{B}_{[6]}^\mu \gamma^\nu \gamma_5 U_\mu B_{[6]} U_\nu^T + \text{h.c.}) \\
& + f_{D,[66]}^{(A)} \text{tr} (\bar{B}_{[6]}^\mu \gamma^\nu \gamma_5 \{U_\mu, U_\nu\} B_{[6]} + \text{h.c.}) + f_{F,[66]}^{(A)} \text{tr} (\bar{B}_{[6]}^\mu \gamma^\nu \gamma_5 [U_\mu, U_\nu] B_{[6]} + \text{h.c.}) \\
& + f_{1,[36]}^{(A)} \text{tr} (\bar{B}_{[6]}^\mu \gamma^\nu \gamma_5 U_\nu B_{[3]} U_\mu^T + \bar{B}_{[6]}^\mu \gamma^\nu \gamma_5 U_\mu B_{[3]} U_\nu^T + \text{h.c.}) \\
& + f_{D,[36]}^{(A)} \text{tr} (\bar{B}_{[6]}^\mu \gamma^\nu \gamma_5 \{U_\mu, U_\nu\} B_{[3]} + \text{h.c.}) + f_{F,[36]}^{(A)} \text{tr} (\bar{B}_{[6]}^\mu \gamma^\nu \gamma_5 [U_\mu, U_\nu] B_{[3]} + \text{h.c.}), \\
\mathcal{L}^{(T)} = & g_{F,[33]}^{(T)} \text{tr} (\bar{B}_{[3]} i \sigma^{\alpha\beta} [U_\alpha, U_\beta] B_{[3]}) + g_{1,[36]}^{(T)} \text{tr} (\bar{B}_{[6]} i \sigma^{\alpha\beta} U_\alpha B_{[3]} U_\beta^T + \text{h.c.}) \\
& + g_{F,[36]}^{(T)} \text{tr} (\bar{B}_{[6]} i \sigma^{\alpha\beta} [U_\alpha, U_\beta] B_{[3]} + \text{h.c.}) + g_{F,[66]}^{(T)} \text{tr} (\bar{B}_{[6]} i \sigma^{\alpha\beta} [U_\alpha, U_\beta] B_{[6]}) \\
& + h_{F,[66]}^{(T)} \text{tr} (\bar{B}_{[6]}^\mu g_{\mu\nu} i \sigma^{\alpha\beta} [U_\alpha, U_\beta] B_{[6]}^\nu),
\end{aligned}$$

where further possible terms that are redundant owing to the on-shell conditions of spin- $\frac{3}{2}$ fields with $\gamma_\mu B_{[6]}^\mu = 0$ and $\partial_\mu B_{[6]}^\mu = 0$ are eliminated systematically.

The symmetry conserving parameters contribute to the current-current correlation function of two time-ordered axial-vector currents

$$A_{\mu\nu}^{ab}(q) = i \int d^4x e^{-iq \cdot x} \mathcal{T} A_\mu^{(a)}(x) A_\nu^{(b)}(0), \quad (15)$$

in the baryon states. The latter can be analyzed systematically in the $1/N_c$ expansion [7, 18].

In order to prepare for a matching we derive the specific form of such contributions

$$\begin{aligned}
\langle \bar{p}, mn_+, \tfrac{1}{2}, \bar{\chi} | A_{ij}^{ab}(q) | p, kl_+, \tfrac{1}{2}, \chi \rangle &= \epsilon_{ijk} \sigma_{\bar{\chi}\chi}^{(k)} g_{F,[66]}^{(T)} f_{abc} \Lambda_{(kl)_+}^{(c),(mn)+} \\
&+ \delta_{ij} \delta_{\bar{\chi}\chi} \frac{1}{2} \left\{ \left(g_{0,[66]}^{(S)} - \frac{1}{3} g_{1,[66]}^{(S)} \right) 2 \delta_{ab} \delta_{(kl)_+}^{(mn)+} + \left(g_{D,[66]}^{(S)} - \frac{1}{2} g_{1,[66]}^{(S)} \right) 2 d_{abc} \Lambda_{(kl)_+}^{(c),(mn)+} \right. \\
&+ g_{1,[66]}^{(S)} \left(\Lambda_{(rs)_+}^{(a),(mn)+} \Lambda_{(kl)_+}^{(b),(rs)+} + \Lambda_{(rs)_+}^{(b),(mn)+} \Lambda_{(kl)_+}^{(a),(rs)+} \right) \Big\} \\
&- \frac{(\bar{p}+p)_i (\bar{p}+p)_j}{M} \delta_{\bar{\chi}\chi} \left\{ \left(g_{0,[66]}^{(V)} - \frac{2}{3} g_{1,[66]}^{(V)} \right) \delta_{ab} \delta_{(kl)_+}^{(mn)+} + \left(g_{D,[66]}^{(V)} - g_{1,[66]}^{(V)} \right) d_{abc} \Lambda_{(kl)_+}^{(c),(mn)+} \right. \\
&+ g_{1,[66]}^{(V)} \left(\Lambda_{(rs)_+}^{(a),(mn)+} \Lambda_{(kl)_+}^{(b),(rs)+} + \Lambda_{(rs)_+}^{(b),(mn)+} \Lambda_{(kl)_+}^{(a),(rs)+} \right) \Big\} + \dots,
\end{aligned}$$

$$\begin{aligned}
\langle \bar{p}, mn_-, \tfrac{1}{2}, \bar{\chi} | A_{ij}^{ab}(q) | p, kl_-, \tfrac{1}{2}, \chi \rangle &= \epsilon_{ijk} \sigma_{\bar{\chi}\chi}^{(k)} g_{F,[\bar{3}\bar{3}]}^{(T)} f_{abc} \Lambda_{(kl)_-}^{(c),(mn)-} \\
&- \frac{(\bar{p}+p)_i (\bar{p}+p)_j}{M} \delta_{\bar{\chi}\chi} \left\{ \left(g_{0,[\bar{3}\bar{3}]}^{(V)} - \frac{2}{3} g_{1,[\bar{3}\bar{3}]}^{(V)} \right) \delta_{ab} \delta_{(kl)_-}^{(mn)-} + \left(g_{D,[\bar{3}\bar{3}]}^{(V)} - g_{1,[\bar{3}\bar{3}]}^{(V)} \right) d_{abc} \Lambda_{(kl)_-}^{(c),(mn)-} \right. \\
&+ g_{1,[\bar{3}\bar{3}]}^{(V)} \left(\Lambda_{(rs)_-}^{(a),(mn)-} \Lambda_{(kl)_-}^{(b),(rs)-} + \Lambda_{(rs)_-}^{(b),(mn)-} \Lambda_{(kl)_-}^{(a),(rs)-} \right) \Big\} \\
&+ \delta_{ij} \delta_{\bar{\chi}\chi} \frac{1}{2} \left\{ 2 g_{0,[\bar{3}\bar{3}]}^{(S)} \delta_{ab} \delta_{(kl)_-}^{(mn)-} + 2 g_{D,[\bar{3}\bar{3}]}^{(S)} d_{abc} \Lambda_{(kl)_-}^{(c),(mn)-} \right\} + \dots,
\end{aligned}$$

$$\begin{aligned}
\langle \bar{p}, mn_-, \tfrac{1}{2}, \bar{\chi} | A_{ij}^{ab}(q) | p, kl_+, \tfrac{1}{2}, \chi \rangle &= \epsilon_{ijk} \sigma_{\bar{\chi}\chi}^{(k)} \left\{ \left(g_{F,[\bar{3}6]}^{(T)} - \frac{1}{2} g_{1,[\bar{3}6]}^{(T)} \right) f_{abc} \Lambda_{(kl)_+}^{(c),(mn)-} \right. \\
&- \frac{i}{2} g_{1,[\bar{3}6]}^{(T)} \left(\Lambda_{(rs)_+}^{(a),(mn)-} \Lambda_{(kl)_+}^{(b),(rs)+} - \Lambda_{(rs)_+}^{(b),(mn)-} \Lambda_{(kl)_+}^{(a),(rs)+} \right) \Big\} \\
&- \frac{(\bar{p}+p)_i (\bar{p}+p)_j}{M} \delta_{\bar{\chi}\chi} \left\{ \left(g_{D,[\bar{3}6]}^{(V)} - g_{1,[\bar{3}6]}^{(V)} \right) d_{abc} \Lambda_{(kl)_+}^{(c),(mn)-} \right. \\
&+ g_{1,[\bar{3}6]}^{(V)} \left(\Lambda_{(rs)_+}^{(a),(mn)-} \Lambda_{(kl)_+}^{(b),(rs)+} + \Lambda_{(rs)_+}^{(b),(mn)-} \Lambda_{(kl)_+}^{(a),(rs)+} \right) \Big\} \\
&+ \delta_{ij} \delta_{\bar{\chi}\chi} \left\{ \left(g_{D,[\bar{3}6]}^{(S)} - \frac{1}{2} g_{1,[\bar{3}6]}^{(S)} \right) d_{abc} \Lambda_{(kl)_+}^{(c),(mn)-} \right. \\
&+ \frac{1}{2} g_{1,[\bar{3}6]}^{(S)} \left(\Lambda_{(rs)_+}^{(a),(mn)-} \Lambda_{(kl)_+}^{(b),(rs)+} + \Lambda_{(rs)_+}^{(b),(mn)-} \Lambda_{(kl)_+}^{(a),(rs)+} \right) \Big\} + \dots,
\end{aligned}$$

$$\begin{aligned}
\langle \bar{p}, mn_+, \tfrac{3}{2}, \bar{\chi} | A_{ij}^{ab}(q) | p, kl_+, \tfrac{1}{2}, \chi \rangle &= i \epsilon_{ijk} S_{\bar{\chi}\chi}^{(k)} \frac{1}{2} f_{F,[66]}^{(A)} i f_{abc} \Lambda_{(kl)_+}^{(c),(mn)+} \\
&- \left(S_i \sigma_j + S_j \sigma_i \right)_{\bar{\chi}\chi} \frac{1}{2} \left\{ \left(f_{0,[66]}^{(A)} - \frac{2}{3} f_{1,[66]}^{(A)} \right) \delta_{ab} \delta_{(kl)_+}^{(mn)+} \right. \\
&+ \left(f_{D,[66]}^{(A)} - f_{1,[66]}^{(A)} \right) d_{abc} \Lambda_{(kl)_+}^{(c),(mn)+} \\
&+ f_{1,[66]}^{(A)} \left(\Lambda_{(rs)_+}^{(a),(mn)+} \Lambda_{(kl)_+}^{(b),(rs)+} + \Lambda_{(rs)_+}^{(b),(mn)+} \Lambda_{(kl)_+}^{(a),(rs)+} \right) \Big\} + \dots,
\end{aligned}$$

$$\langle \bar{p}, mn_+, \tfrac{3}{2}, \bar{\chi} | A_{ij}^{ab}(q) | p, kl_-, \tfrac{1}{2}, \chi \rangle = i \epsilon_{ijk} S_{\bar{\chi}\chi}^{(k)} \frac{1}{2} f_{F,[\bar{3}6]}^{(A)} f_{abc} \Lambda_{(kl)_-}^{(c),(mn)+}$$

$$\begin{aligned}
& - \left(S_i \sigma_j + S_j \sigma_i \right)_{\bar{\chi}\chi} \frac{1}{2} \left\{ \left(f_{D,[36]}^{(A)} - f_{1,[36]}^{(A)} \right) d_{abc} \Lambda_{(kl)-}^{(c),(mn)+} \right. \\
& \quad \left. + f_{1,[36]}^{(A)} \left(\Lambda_{(rs)-}^{(a),(mn)+} \Lambda_{(kl)-}^{(b),(rs)-} + \Lambda_{(rs)-}^{(b),(mn)+} \Lambda_{(kl)-}^{(a),(rs)-} \right) \right\} + \dots,
\end{aligned}$$

$$\begin{aligned}
\langle \bar{p}, mn_+, \frac{3}{2}, \bar{\chi} | A_{ij}^{ab}(q) | p, kl_+, \frac{3}{2}, \chi \rangle = & - \epsilon_{ijk} \left(\vec{S} \sigma^{(k)} \vec{S}^\dagger \right)_{\bar{\chi}\chi} h_{F,[66]}^{(T)} f_{abc} \Lambda_{(kl)+}^{(c),(mn)+} \\
& - \delta_{ij} \delta_{\bar{\chi}\chi} \frac{1}{2} \left\{ \left(h_{0,[66]}^{(S)} - \frac{1}{3} h_{4,[66]}^{(S)} \right) 2 \delta_{ab} \delta_{(kl)+}^{(mn)+} + \left(h_{2,[66]}^{(S)} - \frac{1}{2} h_{4,[66]}^{(S)} \right) 2 d_{abc} \Lambda_{(kl)+}^{(c),(mn)+} \right. \\
& \quad \left. + h_{4,[66]}^{(S)} \left(\Lambda_{(rs)+}^{(a),(mn)+} \Lambda_{(kl)+}^{(b),(rs)+} + \Lambda_{(rs)+}^{(b),(mn)+} \Lambda_{(kl)+}^{(a),(rs)+} \right) \right\} \\
& - \left(S_i S_j^\dagger + S_j S_i^\dagger \right)_{\bar{\chi}\chi} \frac{1}{2} \left\{ \left(h_{1,[66]}^{(S)} - \frac{2}{3} h_{5,[66]}^{(S)} \right) \delta_{ab} \delta_{(kl)+}^{(mn)+} + \left(h_{3,[66]}^{(S)} - h_{5,[66]}^{(S)} \right) d_{abc} \Lambda_{(kl)+}^{(c),(mn)+} \right. \\
& \quad \left. + h_{5,[66]}^{(S)} \left(\Lambda_{(rs)+}^{(a),(mn)+} \Lambda_{(kl)+}^{(b),(rs)+} + \Lambda_{(rs)+}^{(b),(mn)+} \Lambda_{(kl)+}^{(a),(rs)+} \right) \right\} \\
& + \frac{(\bar{p} + p)_i (\bar{p} + p)_j}{M} \delta_{\bar{\chi}\chi} \left\{ \left(h_{0,[66]}^{(V)} - \frac{2}{3} h_{1,[66]}^{(V)} \right) \delta_{ab} \delta_{(kl)+}^{(mn)+} + \left(h_{2,[66]}^{(V)} - h_{1,[66]}^{(V)} \right) d_{abc} \Lambda_{(kl)+}^{(c),(mn)+} \right. \\
& \quad \left. + h_{1,[66]}^{(V)} \left(\Lambda_{(rs)+}^{(a),(mn)+} \Lambda_{(kl)+}^{(b),(rs)+} + \Lambda_{(rs)+}^{(b),(mn)+} \Lambda_{(kl)+}^{(a),(rs)+} \right) \right\} + \dots, \tag{16}
\end{aligned}$$

where $q = \bar{p} - p$ and $a, b = 1, \dots, 8$. For the spin and flavour structures we apply our notations (10, 13). Like in (9) the dots in (16) represent additional terms that are further suppressed for small 3-momenta p and \bar{p} . Here we also assumed a degenerate baryon mass M for the baryon states as they arise in the large- N_c limit. Like in (9) we focus on the spatial components of the axial-vector currents as they suffice to establish the desired correlations of the counter terms that arise in the large- N_c limit.

We close this section with a partial collection of terms contributing to $\mathcal{L}_\chi^{(4)}$ that are relevant in a chiral extrapolation of the baryon masses at N³LO. There are 17 symmetry breaking counter terms

$$\begin{aligned}
\mathcal{L}_\chi^{(4)} = & c_{1,[\bar{3}\bar{3}]} \text{tr} \left(\bar{B}_{[\bar{3}]} B_{[\bar{3}]} \right) \text{tr} \left(\chi_+^2 \right) + c_{2,[\bar{3}\bar{3}]} \text{tr} \left(\bar{B}_{[\bar{3}]} B_{[\bar{3}]} \right) \left(\text{tr} \chi_+ \right)^2 \\
& + c_{3,[\bar{3}\bar{3}]} \text{tr} \left(\bar{B}_{[\bar{3}]} \chi_+ B_{[\bar{3}]} \right) \text{tr} \left(\chi_+ \right) + c_{4,[\bar{3}\bar{3}]} \text{tr} \left(\bar{B}_{[\bar{3}]} \chi_+^2 B_{[\bar{3}]} \right) \\
& + c_{1,[66]} \text{tr} \left(\bar{B}_{[6]} B_{[6]} \right) \text{tr} \left(\chi_+^2 \right) + c_{2,[66]} \text{tr} \left(\bar{B}_{[6]} B_{[6]} \right) \left(\text{tr} \chi_+ \right)^2 \\
& + c_{3,[66]} \text{tr} \left(\bar{B}_{[6]} \chi_+ B_{[6]} \right) \text{tr} \left(\chi_+ \right) + c_{4,[66]} \text{tr} \left(\bar{B}_{[6]} \chi_+^2 B_{[6]} \right) + c_{5,[66]} \text{tr} \left(\bar{B}_{[6]} \chi_+ B_{[6]} \chi_+^T \right) \\
& + c_{1,[\bar{3}6]} \text{tr} \left(\bar{B}_{[6]} \chi_+ B_{[\bar{3}]} + \text{h.c.} \right) \text{tr} \left(\chi_+ \right) + c_{2,[\bar{3}6]} \text{tr} \left(\bar{B}_{[6]} \chi_+^2 B_{[\bar{3}]} + \text{h.c.} \right) \\
& + c_{3,[\bar{3}6]} \text{tr} \left(\bar{B}_{[6]} \chi_+ B_{[\bar{3}]} \chi_+^T + \text{h.c.} \right) \\
& - e_{1,[66]} \text{tr} \left(\bar{B}_{[6]\mu} B_{[6]}^\mu \right) \text{tr} \left(\chi_+^2 \right) - e_{2,[66]} \text{tr} \left(\bar{B}_{[6]\mu} B_{[6]}^\mu \right) \left(\text{tr} \chi_+ \right)^2 \\
& - e_{3,[66]} \text{tr} \left(\bar{B}_{[6]\mu} \chi_+ B_{[6]}^\mu \right) \text{tr} \left(\chi_+ \right) - e_{4,[66]} \text{tr} \left(\bar{B}_{[6]\mu} \chi_+^2 B_{[6]}^\mu \right)
\end{aligned}$$

$$- e_{5,[66]} \text{tr} \left(\bar{B}_{[6]\mu} \chi_+ B_{[6]}^\mu \chi_+ \right). \quad (17)$$

The symmetry breaking counter terms contribute to the current-current correlation function of two time-ordered scalar currents

$$S^{ab}(q) = i \int d^4x e^{-iq \cdot x} \mathcal{T} S^{(a)}(x) S^{(b)}(0), \quad (18)$$

in the baryon states. Like in (12) we consider singlet and octet components with $a, b = 0, \dots, 8$. The latter can be analyzed systematically in the $1/N_c$ expansion [7, 18]. In order to prepare for a matching we derive the specific form of such contributions

$$\begin{aligned} \langle \bar{p}, mn_+, \tfrac{1}{2}, \bar{\chi} | S^{ab}(q) | p, kl_+, \tfrac{1}{2}, \chi \rangle &= \delta_{\bar{\chi}\chi} \frac{1}{2} \left\{ \left(c_{1,[66]} - \frac{1}{3} c_{5,[66]} \right) 2 \delta_{ab} \delta_{(kl)_+}^{(mn)_+} \right. \\ &\quad + \left(c_{4,[66]} - \frac{1}{2} c_{5,[66]} \right) 2 d_{abc} \Lambda_{(kl)_+}^{(c),(mn)_+} \\ &\quad + c_{5,[66]} \left(\Lambda_{(rs)_+}^{(a),(mn)_+} \Lambda_{(kl)_+}^{(b),(rs)_+} + \Lambda_{(rs)_+}^{(b),(mn)_+} \Lambda_{(kl)_+}^{(a),(rs)_+} \right) \\ &\quad \left. + c_{2,[66]} 6 \delta_{a0} \delta_{b0} \delta_{(kl)_+}^{(mn)_+} + c_{3,[66]} \sqrt{\frac{3}{2}} \left(\delta_{a0} \Lambda_{(kl)_+}^{(b),(mn)_+} + \delta_{b0} \Lambda_{(kl)_+}^{(a),(mn)_+} \right) \right\} + \dots, \\ \langle \bar{p}, mn_-, \tfrac{1}{2}, \bar{\chi} | S^{ab}(q) | p, kl_-, \tfrac{1}{2}, \chi \rangle &= \delta_{\bar{\chi}\chi} \frac{1}{2} \left\{ c_{1,[\bar{3}\bar{3}]} 2 \delta_{ab} \delta_{(kl)_-}^{(mn)_-} + c_{4,[\bar{3}\bar{3}]} 2 d_{abc} \Lambda_{(kl)_-}^{(c),(mn)_-} \right. \\ &\quad \left. + c_{2,[\bar{3}\bar{3}]} 6 \delta_{a0} \delta_{b0} \delta_{(kl)_-}^{(mn)_-} + c_{3,[\bar{3}\bar{3}]} \sqrt{\frac{3}{2}} \left(\delta_{a0} \Lambda_{(kl)_-}^{(b),(mn)_-} + \delta_{b0} \Lambda_{(kl)_-}^{(a),(mn)_-} \right) \right\} + \dots, \\ \langle \bar{p}, mn_-, \tfrac{1}{2}, \bar{\chi} | S^{ab}(q) | p, kl_+, \tfrac{1}{2}, \chi \rangle &= \delta_{\bar{\chi}\chi} \frac{1}{2} \left\{ \left(c_{2,[\bar{3}6]} - \frac{1}{2} c_{3,[\bar{3}6]} \right) 2 d_{abc} \Lambda_{(kl)_+}^{(c),(mn)_-} \right. \\ &\quad + c_{3,[\bar{3}6]} \left(\Lambda_{(rs)_-}^{(a),(mn)_-} \Lambda_{(kl)_+}^{(b),(rs)_-} + \Lambda_{(rs)_-}^{(b),(mn)_-} \Lambda_{(kl)_+}^{(a),(rs)_-} \right) \\ &\quad \left. + c_{1,[\bar{3}6]} \sqrt{\frac{3}{2}} \left(\delta_{a0} \Lambda_{(kl)_+}^{(b),(mn)_-} + \delta_{b0} \Lambda_{(kl)_+}^{(a),(mn)_-} \right) \right\} + \dots, \\ \langle \bar{p}, mn_+, \tfrac{3}{2}, \bar{\chi} | S^{ab}(q) | p, kl_{\pm}, \tfrac{1}{2}, \chi \rangle &= 0 + \dots, \\ \langle \bar{p}, mn_+, \tfrac{3}{2}, \bar{\chi} | S^{ab}(q) | p, kl_+, \tfrac{3}{2}, \chi \rangle &= \delta_{\bar{\chi}\chi} \frac{1}{2} \left\{ \left(e_{1,[66]} - \frac{1}{3} e_{5,[66]} \right) 2 \delta_{ab} \delta_{(kl)_+}^{(mn)_+} \right. \\ &\quad + \left(e_{4,[66]} - \frac{1}{2} e_{5,[66]} \right) 2 d_{abc} \Lambda_{(kl)_+}^{(c),(mn)_+} \\ &\quad + e_{5,[66]} \left(\Lambda_{(rs)_+}^{(a),(mn)_+} \Lambda_{(kl)_+}^{(b),(rs)_+} + \Lambda_{(rs)_+}^{(b),(mn)_+} \Lambda_{(kl)_+}^{(a),(rs)_+} \right) \\ &\quad \left. + e_{2,[66]} 6 \delta_{a0} \delta_{b0} \delta_{(kl)_+}^{(mn)_+} + e_{3,[66]} \sqrt{\frac{3}{2}} \left(\delta_{a0} \Lambda_{(kl)_+}^{(b),(mn)_+} + \delta_{b0} \Lambda_{(kl)_+}^{(a),(mn)_+} \right) \right\} + \dots, \quad (19) \end{aligned}$$

where the intermediate flavour index $c = 1, \dots, 8$ is summed over the octet components only. We use the convention

$$d_{0ab} = d_{a0b} = d_{ab0} = \sqrt{\frac{2}{3}} \delta_{ab}. \quad (20)$$

Altogether we introduced $9+7+36+17 = 69$ distinct low-energy constants, which have to be determined. Out of the 69 terms there are 56 distinct terms relevant in a chiral expansion of the baryon masses at N³LO. This may appear a hopeless situation. However, this is not the case due to the heavy-quark spin symmetry and sum rules that arise in the large- N_c limit of QCD.

III. HEAVY QUARK MASS EXPANSION

In the limit of an infinite charm quark mass the two sextet fields can be combined into a super multiplet field [22–24, 29]. This reflects the fact that in this limit the $\frac{1}{2}^+$ and $\frac{3}{2}^+$ baryons are related by a spin flip of the charm quark, which does not cost any energy. Therefore, the properties of such states are closely related. In order to work out the implications of the heavy-quark symmetry of QCD it is useful to introduce auxiliary and slowly varying fields, $B_{\pm}(x)$ and $B_{\pm}^{\mu}(x)$. We decompose the baryon sextet fields into such components

$$\begin{aligned} B_{[6]}(x) &= e^{-i(v \cdot x) M_{[6]}^{1/2}} B_+(x) + e^{+i(v \cdot x) M_{[6]}^{1/2}} B_-(x), \\ B_{[6]}^{\mu}(x) &= e^{-i(v \cdot x) M_{[6]}^{3/2}} B_+^{\mu}(x) + e^{+i(v \cdot x) M_{[6]}^{3/2}} B_-^{\mu}(x), \end{aligned} \quad (21)$$

with a 4-velocity v normalized by $v^2 = 1$. The mass parameter $M_{[6]}^{1/2}$ and $M_{[6]}^{3/2}$ are the chiral limit masses of the two sextet baryons as introduced in (5). A corresponding decomposition for the anti-triplet field $B_{[\bar{3}]}(x)$ is assumed. As a consequence of Eq. (21), time and spatial derivatives of the fields B_{\pm} and B_{\pm}^{μ} are small compared to $M_{[6]} v_{\alpha} B_{\pm}(x)$. In the limit $M_{[6]} \rightarrow \infty$ the former terms can be neglected. Note that the fields $B_{\pm}(x)$ and $B_{\pm}^{\mu}(x)$ annihilate quanta with charm content ± 1 .

The mass parameters $M_{[3]}^{1/2}$, $M_{[6]}^{1/2}$ and $M_{[6]}^{3/2}$ may be expanded in inverse powers of the charm quark mass M_c . A matching with QCD's properties [21–23, 29] leads to the scaling properties

$$M_{[6]}^{3/2} - M_{[6]}^{1/2} \sim \frac{1}{M_c}, \quad M_{[6]}^{1/2} - M_{[3]}^{1/2} \sim 1, \quad (22)$$

which implies that the two sextet masses are degenerate in the heavy-quark mass limit. We apply here the formalism developed in [22–24, 29] and introduce the multiplet field $H_{[6]}^{\mu}$, connected to the fields $B_+(x)$ and $B_+^{\mu}(x)$ in the heavy-quark mass limit as follows¹

$$H_{[6]}^{\mu} = \frac{1}{\sqrt{3}} (\gamma^{\mu} + v^{\mu}) \gamma_5 \frac{1 + \not{v}}{2} B_+ + \frac{1 + \not{v}}{2} B_+^{\mu}, \quad \bar{H}_{[6]}^{\mu} = (H_{[6]}^{\mu})^{\dagger} \gamma_0, \quad (23)$$

According to [23, 29], the field $H_{[6]}^{\mu}$ transforms under the heavy-quark spin symmetry group $SU_v(2)$, the elements of which being characterized by the 4-vector θ^{α} with $\theta \cdot v = 0$, as follows:

$$\begin{aligned} H_{[6]}^{\mu} &\rightarrow e^{-i J_{\alpha} \theta^{\alpha}} H_{[6]}^{\mu}, & \bar{H}_{[6]}^{\mu} &\rightarrow (e^{-i J_{\alpha} \theta^{\alpha}} H_{[6]}^{\mu})^{\dagger} \gamma_0 = \bar{H}_{[6]}^{\mu} e^{+i J_{\alpha} \theta^{\alpha}}, \\ J_{\alpha} &= \frac{1}{2} \gamma_5 [\not{v}, \gamma_{\alpha}], & J_{\alpha}^{\dagger} \gamma_0 &= \gamma_0 J_{\alpha}, & [\not{v}, J_{\alpha}]_- &= 0. \end{aligned} \quad (24)$$

¹ Note that $\text{tr } \gamma_5 \gamma_{\mu} \gamma_{\nu} \gamma_{\alpha} \gamma_{\beta} = -4i \epsilon_{\mu\nu\alpha\beta}$ in the convention used in this work.

In (24) we denote the heavy-quark spin-operator with J_α as to avoid any notational confusion with the spin transition matrices S_i introduced in (10). Under a Lorentz transformation $\Lambda_{\mu\nu}$, characterized by the antisymmetric tensor $\omega_{\mu\nu}$, the spinor part of the field transforms as

$$\begin{aligned} H_{[6]}^\mu &\rightarrow e^{+i J_{\alpha\beta} \omega^{\alpha\beta}} \Lambda_\nu^\mu H_{[6]}^\nu, & J_{\alpha\beta} &= \frac{i}{4} [\gamma_\alpha, \gamma_\beta], \\ \bar{H}_{[6]}^\mu &\rightarrow \Lambda_\nu^\mu \bar{H}_{[6]}^\nu e^{-i J_{\alpha\beta} \omega^{\alpha\beta}}, \end{aligned} \quad (25)$$

An analogous construction is applied for the flavour anti-triplet field

$$\begin{aligned} B_{[\bar{3}]}(x) &= e^{-i(v \cdot x) M_c} B_{[\bar{3}]}^{(+)}(x) + e^{+i(v \cdot x) M_c} B_{[\bar{3}]}^{(-)}(x), \\ H_{[\bar{3}]} &= \frac{1 + \not{v}}{2} B_{[\bar{3}]}^{(+)}, & \bar{H}_{[\bar{3}]} &= (H_{[\bar{3}]})^\dagger \gamma_0. \end{aligned} \quad (26)$$

It follows that only field combinations of the form, where there is a nontrivial Dirac matrix neither left of the fields $H_{[6]}^\mu$ and $H_{\bar{3}}$ nor right to the fields $\bar{H}_{[6]}^\mu$ and $\bar{H}_{[\bar{3}]}$, are invariant under the spin group $SU_v(2)$. With this rule it is straightforward to construct $SU_v(2)$ -invariant interaction terms. There is one exception to that rule. The γ_5 matrix commutes with J_α in (24) and therefore γ_5 does not violate the spin symmetry. However such terms vanish due to $\not{v} H = H$ and the identity

$$\frac{1 + \not{v}}{2} \gamma_5 \frac{1 + \not{v}}{2} = 0. \quad (27)$$

Moreover the matrix γ_μ does not commute with J_α . Nevertheless it sometimes leads to spin symmetric terms. This is a consequence of the projection result

$$\frac{1 + \not{v}}{2} \gamma_\mu \frac{1 + \not{v}}{2} = v_\mu \frac{1 + \not{v}}{2}, \quad (28)$$

which, however, also shows that terms with γ_μ will not lead to further spin symmetric terms.

In a first step we reproduce previous results obtained in [21–23]. There are two spin-symmetric kinetic terms

$$\mathcal{L}_{\text{kinetic}}^{(H)} = \text{tr} \bar{H}_{[\bar{3}]} g_{\mu\nu} v^\mu i D^\nu H_{[\bar{3}]} - \text{tr} \bar{H}_{[6]}^\mu g_{\mu\nu} (i D \cdot v) H_{[6]}^\nu. \quad (29)$$

This implies that the two sextet masses $M_{[6]}^{1/2}$ and $M_{[6]}^{3/2}$ are degenerate in the heavy-quark mass limit [21–23]. According to (22) the anti-triplet mass $M_{[\bar{3}]}^{1/2}$ remains as an independent parameter in that limit. We continue with the constraint on the three-point vertices, for which the spin-symmetric interactions

$$\mathcal{L}_{3\text{-point}}^{(H)} = i F_1 \epsilon_{\mu\nu\alpha\beta} \text{tr} \bar{H}_{[6]}^\mu v^\alpha i U^\beta H_{[6]}^\nu + F_2 \text{tr} (\bar{H}_{[6]}^\mu i U_\mu H_{[\bar{3}]} + \text{h.c.}), \quad (30)$$

$g_{0,[\bar{3}\bar{3}]}^{(S)}$	h_2	$-\frac{2}{3}g_1 - \frac{1}{3}g_2 + \frac{1}{12}g_+$	$g_{0,[\bar{3}\bar{3}]}^{(V)}$	h_4	$-\frac{2}{3}g_3 - \frac{1}{6}g_4$
$g_{F,[\bar{3}\bar{3}]}^{(T)}$	0	0	$g_{1,[\bar{3}\bar{3}]}^{(V)}$	0	$g_3 - \frac{1}{2}g_4$
$g_{D,[\bar{3}\bar{3}]}^{(S)}$	h_1	$-g_1 - \frac{1}{2}g_2 + \frac{1}{8}g_+$	$g_{D,[\bar{3}\bar{3}]}^{(V)}$	h_3	$-g_3 - \frac{1}{4}g_4$
$g_{0,[66]}^{(S)}$	$-h_6 - \frac{1}{3}h_{13}$	$-\frac{2}{3}g_1 - \frac{1}{3}g_2 + \frac{1}{12}g_+$	$g_{0,[66]}^{(V)}$	$\frac{1}{3}h_{13} - \frac{1}{3}h_9$	$-\frac{2}{3}g_3 - \frac{1}{6}g_4$
$g_{1,[66]}^{(S)}$	$-h_7 - \frac{2}{3}h_{14}$	$2g_1 - g_2 + \frac{1}{6}g_+$	$g_{1,[66]}^{(V)}$	$-h_{10}$	$g_3 - \frac{1}{2}g_4$
$g_{D,[66]}^{(S)}$	$-h_5 - \frac{1}{3}h_{11}$	$-g_1 - \frac{1}{2}g_2 + \frac{1}{8}g_+$	$g_{D,[66]}^{(V)}$	$\frac{1}{3}h_{11} - \frac{1}{3}h_8$	$-g_3 - \frac{1}{4}g_4$
$g_{F,[36]}^{(T)}$	$\frac{1}{\sqrt{3}}h_{15}$	$-\frac{1}{2\sqrt{3}}g_5 - \frac{1}{8\sqrt{3}}g_-$	$g_{F,[36]}^{(T)}$	$\frac{1}{3}h_{12}$	$\frac{1}{3}g_5 + \frac{1}{24}g_-$
$g_{1,[36]}^{(T)}$	$\frac{1}{\sqrt{3}}h_{16}$	$\frac{1}{\sqrt{3}}g_5 + \frac{1}{4\sqrt{3}}g_-$	$f_{1,[36]}^{(A)}$	0	0
$g_{1,[36]}^{(V)}$	0	0	$f_{D,[36]}^{(A)}$	0	0
$g_{D,[36]}^{(V)}$	0	0	$f_{F,[36]}^{(A)}$	$-2h_{15}$	$g_5 + \frac{1}{4}g_-$
$g_{D,[36]}^{(S)}$	0	0	$h_{0,[66]}^{(V)}$	$\frac{1}{2}h_9$	$\frac{2}{3}g_3 + \frac{1}{6}g_4$
$h_{0,[66]}^{(S)}$	h_6	$\frac{2}{3}g_1 + \frac{1}{3}g_2 - \frac{1}{9}g_+$	$h_{1,[66]}^{(V)}$	h_{10}	$-g_3 + \frac{1}{2}g_4$
$h_{1,[66]}^{(S)}$	h_{13}	$\frac{1}{12}g_+$	$h_{2,[66]}^{(V)}$	$\frac{1}{2}h_8$	$g_3 + \frac{1}{4}g_4$
$h_{2,[66]}^{(S)}$	h_5	$g_1 + \frac{1}{2}g_2 - \frac{1}{6}g_+$	$f_{0,[66]}^{(A)}$	$\frac{1}{\sqrt{3}}h_{13}$	$\frac{1}{12\sqrt{3}}g_+$
$h_{3,[66]}^{(S)}$	h_{11}	$\frac{1}{8}g_+$	$f_{1,[66]}^{(A)}$	$\frac{1}{\sqrt{3}}h_{14}$	$\frac{1}{8\sqrt{3}}g_+$
$h_{4,[66]}^{(S)}$	h_7	$-2g_1 + g_2 + \frac{1}{4}g_+$	$f_{D,[66]}^{(A)}$	$\frac{1}{\sqrt{3}}h_{11}$	$\frac{1}{8\sqrt{3}}g_+$
$h_{5,[66]}^{(S)}$	h_{14}	$\frac{1}{8}g_+$	$f_{F,[66]}^{(A)}$	$\frac{1}{\sqrt{3}}h_{12}$	$\frac{1}{\sqrt{3}}g_5 + \frac{1}{4\sqrt{3}}g_-$
$h_{F,[66]}^{(T)}$	0	$-\frac{1}{2}g_5 - \frac{1}{16}g_-$	$g_{1,[36]}^{(S)}$	0	0

TABLE I. The symmetry conserving two-body counter terms as introduced in (14) are correlated by the heavy-quark symmetry (35) and the leading large- N_c operators (57) with $g_{\pm} = g_6 \pm g_7$.

are parameterized by two coupling constants F_1 and F_2 . A matching of the Lagrangian (30) with the corresponding terms in (5) leads to 4 sum rules

$$F_1 = H_{[66]} = \sqrt{3}C_{[66]} = \frac{3}{2}F_{[66]}, \quad F_2 = C_{[36]} = -\sqrt{3}F_{[36]}, \quad F_{[\bar{3}\bar{3}]} = 0, \quad (31)$$

with which we recover the results of [22, 23].

We turn to the symmetry breaking counter terms introduced in (14, 17), for which we

construct their spin-symmetric correspondence

$$\begin{aligned}
\mathcal{L}_\chi^{(H)} = & d_1 \operatorname{tr} (\bar{H}_{[3]} H_{[3]}) \operatorname{tr} (\chi_+) + d_2 \operatorname{tr} (\bar{H}_{[3]} \chi_+ H_{[3]}) \\
& + d_3 \operatorname{tr} (\bar{H}_{[6]}^\mu g_{\mu\nu} H_{[6]}^\nu) \operatorname{tr} (\chi_+) + d_4 \operatorname{tr} (\bar{H}_{[6]}^\mu g_{\mu\nu} \chi_+ H_{[6]}^\nu) \\
& + e_1 \operatorname{tr} (\bar{H}_{[6]}^\mu g_{\mu\nu} H_{[6]}^\nu) \operatorname{tr} (\chi_+^2) + e_2 \operatorname{tr} (\bar{H}_{[6]}^\mu g_{\mu\nu} H_{[6]}^\nu) (\operatorname{tr} \chi_+)^2 \\
& + e_3 \operatorname{tr} (\bar{H}_{[6]}^\mu g_{\mu\nu} \chi_+ H_{[6]}^\nu) \operatorname{tr} (\chi_+) + e_4 \operatorname{tr} (\bar{H}_{[6]}^\mu g_{\mu\nu} \chi_+^2 H_{[6]}^\nu) \\
& + e_5 \operatorname{tr} (\bar{H}_{[6]}^\mu g_{\mu\nu} \chi_+ H_{[6]}^\nu \chi_+^T) + e_6 \operatorname{tr} (\bar{H}_{[3]} H_{[3]}) \operatorname{tr} (\chi_+^2) \\
& + e_7 \operatorname{tr} (\bar{H}_{[3]} H_{[3]}) (\operatorname{tr} \chi_+)^2 + e_8 \operatorname{tr} (\bar{H}_{[3]} \chi_+ H_{[3]}) \operatorname{tr} (\chi_+) \\
& + e_9 \operatorname{tr} (\bar{H}_{[3]} \chi_+^2 H_{[3]}) .
\end{aligned} \tag{32}$$

There are 4 parameters, d_n , relevant at NLO and 9 parameters, e_n , at N³LO in a computation of the baryon masses. A matching of (32) with (14, 17) determines 11 sum rules. There are 3 sum rules for the NLO parameters

$$\begin{aligned}
b_{1,[66]} = d_{1,[66]} = d_3, & \quad b_{2,[66]} = d_{2,[66]} = d_4, & \quad b_{1,[\bar{3}6]} = 0, \\
b_{1,[\bar{3}\bar{3}]} = d_1, & \quad b_{2,[\bar{3}\bar{3}]} = d_2, &
\end{aligned} \tag{33}$$

and 8 sum rules for the N³LO parameters

$$\begin{aligned}
c_{n,[66]} = e_{n,[66]} = e_n & \quad \text{for} \quad n = 1, \dots, 5, \\
c_{n,[\bar{3}6]} = 0 & \quad \text{for} \quad n = 1, \dots, 3, \\
c_{1,[\bar{3}\bar{3}]} = e_6, & \quad c_{2,[\bar{3}\bar{3}]} = e_7, & \quad c_{3,[\bar{3}\bar{3}]} = e_8, & \quad c_{4,[\bar{3}\bar{3}]} = e_9.
\end{aligned} \tag{34}$$

We close this section with our analysis of the 4-point vertices in the chiral Lagrangian as constructed in (14). Altogether there are 16 spin-symmetric terms

$$\begin{aligned}
\mathcal{L}_{4\text{-point}}^H = & \operatorname{tr} \bar{H}_{[3]} \left\{ h_1 \{U_\mu, U_\nu\} H_{[3]} + h_2 \operatorname{tr} (U_\mu U_\nu) H_{[3]} \right\} g^{\mu\nu} \\
& + \operatorname{tr} \bar{H}_{[3]} \left\{ h_3 \{U_\mu, U_\nu\} H_{[3]} + h_4 H_{[3]} \operatorname{tr} (U_\mu U_\nu) \right\} v^\mu v^\nu \\
& + \operatorname{tr} \bar{H}_{[6]}^\alpha \left\{ h_5 \{U_\mu, U_\mu\} H_{[6]}^\beta + h_6 H_{[6]}^\beta \operatorname{tr} (U_\mu U_\nu) + h_7 U_\mu H_{[6]}^\beta U_\nu^T \right\} g_{\alpha\beta} g^{\mu\nu} \\
& + \operatorname{tr} \bar{H}_{[6]}^\alpha \left\{ h_8 \{U_\mu, U_\nu\} H_{[6]}^\beta + h_9 H_{[6]}^\beta \operatorname{tr} (U_\mu U_\nu) + h_{10} U_\mu H_{[6]}^\beta U_\nu^T \right\} g_{\alpha\beta} v^\mu v^\nu \\
& + \operatorname{tr} \bar{H}_{[6]}^\alpha \left\{ h_{11} \{U_\alpha, U_\beta\} + h_{12} [U_\alpha, U_\beta] + h_{13} \operatorname{tr} (U_\alpha U_\beta) \right\} H_{[6]}^\beta \\
& + \operatorname{tr} \bar{H}_{[6]}^\alpha \left\{ h_{14} \left(U_\mu H_{[6]}^\mu U_\alpha^T + U_\alpha H_{[6]}^\mu U_\mu^T \right) \right\} \\
& + \operatorname{tr} \bar{H}_{[6]}^\alpha \left\{ h_{15} [U_\mu, U_\nu] H_{[3]} + h_{16} U_\mu H_{[3]} U_\nu^T \right\} i \epsilon_{\alpha\beta}^{\mu\nu} v^\beta + \text{h.c.},
\end{aligned} \tag{35}$$

where the conceivable two additional terms

$$\text{tr} \left(\bar{H}_{[6]}^\alpha [U_\mu, U_\nu] H_{[6]}^\beta \right) \epsilon_{\alpha\beta}^{\mu\nu}, \quad \text{tr} \left(\bar{H}_{[6]}^\alpha U_\mu H_{[6]}^\beta U_\nu^T \right) \epsilon_{\alpha\beta}^{\mu\nu}, \quad (36)$$

violate parity conservation. The result of a matching of the chiral Lagrangian (14) with (35) is shown in Tab. I. This leads to $36 - 16 = 20$ sum rules

$$\begin{aligned} g_{0,[66]}^{(S)} &= -h_{0,[66]}^{(S)} - \frac{1}{3} h_{1,[66]}^{(S)}, & g_{1,[66]}^{(S)} &= -h_{4,[66]}^{(S)} - \frac{2}{3} h_{5,[66]}^{(S)}, \\ g_{D,[66]}^{(S)} &= -h_{2,[66]}^{(S)} - \frac{1}{3} h_{3,[66]}^{(S)}, & g_{1,[36]}^{(S)} &= 0, & g_{D,[36]}^{(S)} &= 0, \\ g_{0,[66]}^{(V)} &= \frac{1}{3} h_{1,[66]}^{(S)} - \frac{2}{3} h_{0,[66]}^{(V)}, & g_{1,[66]}^{(V)} &= -h_{1,[66]}^{(V)}, & g_{1,[33]}^{(V)} &= 0, \\ g_{D,[66]}^{(V)} &= \frac{1}{3} h_{3,[66]}^{(S)} - \frac{2}{3} h_{2,[66]}^{(V)}, & g_{1,[36]}^{(V)} &= 0, & g_{D,[36]}^{(V)} &= 0, \\ f_{0,[66]}^{(A)} &= \frac{1}{\sqrt{3}} h_{1,[66]}^{(S)}, & f_{1,[66]}^{(A)} &= \frac{1}{\sqrt{3}} h_{5,[66]}^{(S)}, & f_{D,[66]}^{(A)} &= \frac{1}{\sqrt{3}} h_{3,[66]}^{(S)}, \\ g_{F,[66]}^{(T)} &= \frac{1}{\sqrt{3}} f_{F,[66]}^{(A)}, & h_{F,[66]}^{(T)} &= 0, & g_{F,[36]}^{(T)} &= -\frac{1}{2\sqrt{3}} f_{F,[36]}^{(A)}, \\ g_{F,[33]}^{(T)} &= 0, & f_{1,[36]}^{(A)} &= 0, & f_{D,[36]}^{(A)} &= 0. \end{aligned} \quad (37)$$

Altogether the $3 + 6 + 7 + 36 + 17 = 69$ distinct low-energy constants were correlated by $1 + 4 + 3 + 20 + 8 = 36$ sum rules that follow from the heavy-quark spin symmetry. That leaves 33 unknown parameters. It remains to work out additional sum-rules that arise in the large- N_c limit of QCD.

IV. LARGE- N_c OPERATOR ANALYSIS

In this section we further correlate the parameters of the effective interaction introduced in Section II. We follow the works of Luty and March-Russell [19] and of Dashen, Jenkins and Manohar [20, 21]. These works introduced a formalism for a systematic expansion of baryon matrix elements of QCD quark operators in powers of $1/N_c$. In previous works applications of this formalism to correlation functions involving a product of two axial-vector or vector quark currents were worked out [7, 18].

The $1/N_c$ expansion of an l -body QCD operator takes the generic form

$$\mathcal{O}_{QCD}^{(l)} = N_c^l \sum_{m=0}^{N_l} \sum_{n=0}^{N_h} c^{(m+n)} \frac{1}{N_c^{m+n}} \mathcal{O}_l^{(m)} \mathcal{O}_h^{(n)}, \quad (38)$$

where $\mathcal{O}_l^{(m)}$ and $\mathcal{O}_h^{(n)}$ are suitable m - and n -body operators composed of light and heavy-quark field components respectively. In (38) it is assumed that the operators act on baryon states containing N_l and N_h light and heavy quarks with $N_c = N_l + N_h$.

It was shown in [18–21] that the operator expansion (38) can be performed most efficiently in terms of a complete set of static and color-neutral one-body operators that act on effective baryon states rather than the physical states. While the operator on the left-hand side of (38) is to be evaluated in the physical states, the right-hand side was rewritten such that only effective static baryon states occur. In our case the physical and effective baryon states

$$|p, ij_{\pm}, S, \chi\rangle, \quad |ij_{\pm}, S, \chi\rangle, \quad (39)$$

are specified by the momentum p and the flavor indices $i, j, k = 1, 2, 3$. The spin S and the spin-polarization are $\chi = 1, 2$ for the spin one-half ($S = 1/2$) and $\chi = 1, \dots, 4$ for the spin three-half states ($S = 3/2$). The flavour sextet and the anti-triplet are discriminated by their symmetric (index $+$) and anti-symmetric (index $-$) behaviour under the exchange of $i \leftrightarrow j$. At leading order in the $1/N_c$ expansion all considered baryon states are mass degenerate. It is important to note that unlike the physical baryon states, the effective baryon states do not depend on the momentum p . All dynamical information is moved into the coefficient functions.

The effective baryon states $|ij_{\pm}, \chi\rangle$ were shown to have a mean-field structure that can be generated in terms of effective quark operators q and Q for the light and heavy species respectively. A corresponding complete set of color-neutral one-body operators was constructed

in terms of the very same static quark operators

$$\begin{aligned}
\mathbb{1} &= q^\dagger (\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}) q, & J_i &= q^\dagger \left(\frac{\sigma_i}{2} \otimes \mathbf{1} \otimes \mathbf{1} \right) q, \\
T^a &= q^\dagger \left(\mathbf{1} \otimes \frac{\lambda_a}{2} \otimes \mathbf{1} \right) q, & G_i^a &= q^\dagger \left(\frac{\sigma_i}{2} \otimes \frac{\lambda_a}{2} \otimes \mathbf{1} \right) q, \\
\mathbb{1}_Q &= Q^\dagger (\mathbf{1} \otimes \mathbf{1}) Q, & J_Q^{(i)} &= Q^\dagger \left(\frac{\sigma^{(i)}}{2} \otimes \mathbf{1} \right) Q,
\end{aligned} \tag{40}$$

with the quark operators $q = (u, d, s)^t$ and $Q = c$ of the up, down, strange and charm quarks. With λ_a we denote the Gell-Mann matrices supplemented with a singlet matrix $\lambda_0 = \sqrt{2/3} \mathbf{1}$. Here we use a redundant notation with

$$T^0 = \sqrt{\frac{1}{6}} \mathbb{1}, \quad G_i^0 = \sqrt{\frac{1}{6}} J_i, \tag{41}$$

which will turn useful when analyzing matrix elements of scalar currents.

All what is needed in any practical application of the $1/N_c$ expansion is the action of any of the one-body operators introduced in (40) on the effective mean-field type baryon states $|ij_\pm, \chi\rangle$. In fact it suffices to provide results at the physical value $N_c = 3$, for which we generated the following complete list

$$\begin{aligned}
\mathbb{1} |ij_\pm, \tfrac{1}{2}, \chi\rangle &= 2 |ij_\pm, \tfrac{1}{2}, \chi\rangle, & \mathbb{1} |ij_+, \tfrac{3}{2}, \chi\rangle &= 2 |ij_+, \tfrac{3}{2}, \chi\rangle, \\
\mathbb{1}_Q |ij_\pm, \tfrac{1}{2}, \chi\rangle &= |ij_\pm, \tfrac{1}{2}, \chi\rangle, & \mathbb{1}_Q |ij_+, \tfrac{3}{2}, \chi\rangle &= |ij_+, \tfrac{3}{2}, \chi\rangle, \\
T^a |ij_\pm, \tfrac{1}{2}, \chi\rangle &= \Lambda_{(ij)_\pm}^{(a), (mn)\pm} |mn_\pm, \tfrac{1}{2}, \chi\rangle, & T^a |ij_+, \tfrac{3}{2}, \chi\rangle &= \Lambda_{(ij)_+}^{(a), (mn)+} |mn_+, \tfrac{3}{2}, \chi\rangle, \\
J_Q^{(k)} |ij_\pm, \tfrac{1}{2}, \chi\rangle &= \tfrac{1}{2} \sigma_{\bar{\chi}\chi}^{(k)} |ij_\pm, \tfrac{1}{2}, \bar{\chi}\rangle, & J_Q^{(k)} |ij_+, \tfrac{3}{2}, \chi\rangle &= \tfrac{1}{2} (\vec{S} \sigma^{(k)} \vec{S}^\dagger)_{\bar{\chi}\chi} |ij_+, \tfrac{3}{2}, \bar{\chi}\rangle,
\end{aligned}$$

$$\begin{aligned}
J^k |ij_+, \tfrac{1}{2}, \chi\rangle &= \tfrac{2}{3} \sigma_{\bar{\chi}\chi}^{(k)} |ij_+, \tfrac{1}{2}, \bar{\chi}\rangle - \frac{1}{\sqrt{3}} S_{\bar{\chi}\chi}^{(k)} |ij_+, \tfrac{3}{2}, \bar{\chi}\rangle, \\
G_k^a |ij_+, \tfrac{1}{2}, \chi\rangle &= \frac{1}{3} \Lambda_{(ij)_+}^{(a), (mn)+} \sigma_{\bar{\chi}\chi}^{(k)} |mn_+, \tfrac{1}{2}, \bar{\chi}\rangle - \frac{1}{2\sqrt{3}} \Lambda_{(ij)_+}^{(a), (mn)+} S_{\bar{\chi}\chi}^{(k)} |mn_+, \tfrac{3}{2}, \bar{\chi}\rangle \\
&\quad - \frac{1}{2\sqrt{3}} \Lambda_{(ij)_+}^{(a), (mn)-} \sigma_{\bar{\chi}\chi}^{(k)} |mn_-, \tfrac{1}{2}, \bar{\chi}\rangle,
\end{aligned}$$

$$\begin{aligned}
J^k |ij_+, \tfrac{3}{2}, \chi\rangle &= (\vec{S} \sigma^{(k)} \vec{S}^\dagger)_{\bar{\chi}\chi} |ij_+, \tfrac{3}{2}, \bar{\chi}\rangle - \frac{1}{\sqrt{3}} S_{\bar{\chi}\chi}^{(k)\dagger} |ij_+, \tfrac{1}{2}, \bar{\chi}\rangle \\
G_k^a |ij_+, \tfrac{3}{2}, \chi\rangle &= \frac{1}{2} \Lambda_{(ij)_+}^{(a), (mn)+} (\vec{S} \sigma^{(k)} \vec{S}^\dagger)_{\bar{\chi}\chi} |mn_+, \tfrac{3}{2}, \bar{\chi}\rangle - \frac{1}{2\sqrt{3}} \Lambda_{(ij)_+}^{(a), (mn)+} S_{\bar{\chi}\chi}^{(k)\dagger} |mn_+, \tfrac{1}{2}, \bar{\chi}\rangle \\
&\quad - \frac{1}{2} \Lambda_{(ij)_+}^{(a), (mn)-} S_{\bar{\chi}\chi}^{(k)\dagger} |mn_-, \tfrac{1}{2}, \bar{\chi}\rangle,
\end{aligned}$$

$$J^k |ij_-, \frac{1}{2}, \chi) = 0, \\ G_k^a |ij_-, \frac{1}{2}, \chi) = -\frac{1}{2\sqrt{3}} \Lambda_{(ij)_-}^{(a), (mn)+} \sigma_{\bar{\chi}\chi}^{(k)} |mn_+, \frac{1}{2}, \bar{\chi}) - \frac{1}{2} \Lambda_{(ij)_-}^{(a), (mn)+} S_{\bar{\chi}\chi}^{(k)} |mn_+, \frac{3}{2}, \bar{\chi}), \quad (42)$$

in terms of the notation introduced already in (10, 13). Note that all results (42) are valid also for the singlet components with $a = 0$ if the convention $\lambda_0 = \sqrt{2/3} \mathbf{1}$ is used in (10).

In the sum of (38) there are infinitely many terms one may write down. Terms that break the heavy-quark spin symmetry are exclusively caused by the heavy-spin operator

$$J_Q^i \sim \frac{1}{M_Q}, \quad (43)$$

with the heavy-quark mass M_Q . In contrast the counting of N_c factors is intricate since there is a subtle balance of suppression and enhancement effects. An r -body operator consisting of the r products of any of the spin and flavor operators receives the suppression factor N_c^{-r} . This is counteracted by enhancement factors for the flavor and spin-flavor operators T^a and G_i^a that are produced by taking baryon matrix elements at $N_c \neq 3$. Altogether this leads to the effective scaling laws [20, 21]

$$J_i \sim \frac{1}{N_c}, \quad T^a \sim N_c^0, \quad G_i^a \sim N_c^0. \quad (44)$$

According to (44) there is an infinite number of terms contributing at a given order in the $1/N_c$ expansion. Taking higher products of flavor and spin-flavor operators does not reduce the N_c scaling power. A systematic $1/N_c$ expansion is made possible by a set of operator identities [18, 20, 21], that allows a systematic summation of the infinite number of relevant terms.

The reduction algorithm of [18, 20, 21] relies on an analysis of the product of two one-body-operators. First, antisymmetric products are considered. They can always be reduced to a one-body operator by using the Lie-algebra relations

$$\begin{aligned} [J_i, J_j] &= i \varepsilon_{ijk} J_k, & [T^a, T^b] &= i f_{abc} T^c, & [J^i, T^a] &= 0, \\ [J_i, G_j^a] &= i \varepsilon_{ijk} G_k^a, & [T^a, G_i^b] &= i f_{abc} G_i^c, \\ [G_i^a, G_j^b] &= \frac{i}{4} \delta_{ij} f_{abc} T^c + \frac{i}{6} \delta_{ab} \varepsilon_{ijk} J_k + \frac{i}{2} \varepsilon_{ijk} d_{abc} G_k^c, \\ [J_Q^{(i)}, J_Q^{(j)}] &= i \varepsilon_{ijk} J_Q^{(k)}, & [J_Q^{(i)}, J_i] &= [J_Q^{(i)}, T^a] = [J_Q^{(i)}, G_i^a] = 0, \end{aligned} \quad (45)$$

which may be verified by an explicit computation as a consequence of the canonical anti-commutator relations of the static quark operators.

It remains to consider the symmetric products of two one-body operators, which were studied in great depth in [18, 20]. A simplex example of such identities is the anti-commutator of two heavy-spin operators

$$\{J_Q^i, J_Q^j\} = \frac{3}{2} \mathbb{1}_Q. \quad (46)$$

While all Lie-algebra identities (45) hold as such, the identity (46) holds only if matrix elements in the charmed baryon states introduced in (39) are taken. As was observed by Jenkins [21] the relations obtained in [20] can be adapted to the present case of matrix elements in charmed baryons. Formally they follow from the results [18, 20] upon the replacement $N_c \rightarrow N_l$. The complete set of operator identities reads

$$\begin{aligned} \delta_{ab} \{T^a, T^b\} &= \frac{1}{6} (N_l + 6) \mathbb{1} + \{J_k, J_k\}, \\ d_{abc} \{T^a, T^b\} &= -\frac{1}{3} (N_l + 3) T^c + 2 \{J^i, G^{ic}\}, \end{aligned} \quad (47)$$

and

$$\begin{aligned} \delta_{ab} \{T^a, G_i^b\} &= \frac{2}{3} (N_l + 3) J_i, \\ d_{abc} \{T^a, G_i^b\} &= \frac{1}{3} (N_l + 3) G_i^c + \frac{1}{6} \{J_i, T^c\}, \\ f_{abc} \{T^a, G_i^b\} &= \varepsilon_{ijk} \{J_j, G_k^c\}, \end{aligned} \quad (48)$$

and

$$\begin{aligned} \delta_{ab} \{G_i^a, G_j^b\} &= \frac{1}{8} \delta_{ij} \left((N_l + 6) \mathbb{1} - 2 \{J_k, J_k\} \right) + \frac{1}{3} \{J_i, J_j\}, \\ d_{abc} \{G_i^a, G_j^b\} &= \frac{1}{3} \delta_{ij} \left(\frac{4}{3} (N_l + 3) T^c - \frac{3}{2} \{J_k, G_k^c\} \right) \\ &\quad + \frac{1}{6} \left(\{J_i, G_j^c\} + \{J_j, G_i^c\} \right), \\ \{G_k^a, G_k^b\} &= \frac{1}{24} \delta_{ab} \left((N_l + 6) \mathbb{1} - 2 \{J_k, J_k\} \right) \\ &\quad + \frac{1}{2} d_{abc} \left((N_l + 3) T^c - 2 \{J_k, G_k^c\} \right) + \frac{1}{4} \{T^a, T^b\}, \\ \varepsilon_{ijk} \{G_j^a, G_k^b\} &= \frac{1}{2} f_{abc} \left(- (N_l + 3) G_i^c + \frac{1}{6} \{J_i, T^c\} \right) \\ &\quad + \frac{1}{2} \left(f_{acg} d_{bch} - f_{bcg} d_{ach} \right) \{T^g, G_i^h\}, \end{aligned} \quad (49)$$

which hold in matrix elements of the baryon ground-state tower. All identities are confirmed by taking matrix elements in the charmed baryon states with $N_l = 2$. Unlike in (42) all

flavour indices in (47-49) exclude the singlet component. Detailed expressions are collected in the Appendix.

Owing to the operator identities the general two reduction rules of [20] are recovered:

- All operator products in which two flavor indices are contracted using δ_{ab} , f_{abc} or d_{abc} or two spin indices on G 's are contracted using δ_{ij} or ε_{ijk} can be eliminated.
- All operator products in which two flavor indices are contracted using symmetric or antisymmetric combinations of two different d and/or f symbols can be eliminated.

The only exception to this rule is the antisymmetric combination $f_{acg} d_{bch} - f_{bcg} d_{ach}$.

As a consequence the infinite tower of spin-flavor operators truncates at any given order in the $1/N_c$ expansion. We can now turn to the $1/N_c$ expansion of the baryon matrix elements of the QCD's axial-vector and scalar currents. In application of the operator reduction rules, the baryon matrix elements of time-ordered products of the current operators are expanded in powers of the effective one-body operators according to the counting rule (43, 44) supplemented by the reduction rules (45, 46, 47, 48, 49). In contrast to Jenkins [21] we consider the ratio $N_l/N_c = 1 - 1/N_c$ not as a suppression factor. The strength of the spin-symmetry breaking terms we estimate with $1/M_Q \sim 1/N_c$. In the course of the construction of the various structures, parity and time-reversal transformation properties are taken into account.

A first simple application of the operator reduction rules follows with matrix elements of the axial-vector current (6). Keeping terms that are leading and subleading in the $1/N_c$ expansion there are three relevant operators only

$$\begin{aligned} \langle \bar{p}, mn_{\pm}, \bar{S}, \bar{\chi} | A_i^{(a)}(0) | p, kl_{\pm}, S, \chi \rangle = & (mn_{\pm}, \bar{S}, \bar{\chi} | f_1 G_i^a + f_2 \{J_i, T^a\} \\ & + f_3 \{J_Q^i, T^a\} | kl_{\pm}, S, \chi) + \dots \end{aligned} \quad (50)$$

The two spin-symmetric structures are parameterized by $f_{1,2}$ and the one spin-symmetry breaking term with f_3 . Since the last term is suppressed in the heavy-quark mass limit we expect the third structure to be of minor importance only. We would include it at a level where 3-body operators that are further suppressed in the $1/N_c$ expansion are considered. It is instructive to match the three coupling constants to the 6 parameters introduced in (5). A comparison of (9) with matrix elements of (50) as provided in the Appendix leads to

the following identification

$$\begin{aligned}
F_{[66]} &= \frac{1}{3} f_1 + \frac{4}{3} f_2 + f_3, & F_{[\bar{3}3]} &= f_3, & F_{[\bar{3}6]} &= -\frac{1}{2\sqrt{3}} f_1, \\
C_{[66]} &= \frac{1}{2\sqrt{3}} f_1 + \frac{2}{\sqrt{3}} f_2, & C_{[\bar{3}6]} &= \frac{1}{2} f_1, \\
H_{[66]} &= \frac{1}{2} f_1 + 2 f_2 + f_3.
\end{aligned} \tag{51}$$

It is reassuring that the spin-symmetry sum rules (31) are recovered only and only with $f_3 = 0$, which is consistent with the scaling ansatz (43).

We turn to matrix elements of the scalar current (6, 12). At N²LO in the $1/N_c$ expansion there are 5 operators to be considered

$$\begin{aligned}
\langle \bar{p}, mn_{\pm}, \bar{S}, \bar{\chi} | S^{(a)}(0) | p, kl_{\pm}, S, \chi \rangle &= (mn_{\pm}, \bar{S}, \bar{\chi} | \delta_{a0} (b_1 \mathbb{1} + b_2 J^2) \\
&+ b_3 T^a + b_4 \{J^i, G_i^a\} + b_5 \{J_Q^i, G_i^a\} | kl_{\pm}, S, \chi) + \dots.
\end{aligned} \tag{52}$$

Here we consider the spin-symmetry breaking operator $\{J_Q^i, G_i^a\}$ since our counting suggests that it is as important as the J^2 operator. The flavour index a in (52) may take the singlet value $a = 0$ with the singlet operators assumed in the notation (41). The matrix elements of the operators in the baryon states are readily looked up in the Appendix. A comparison with the tree-level expressions (12) generates the identifications

$$\begin{aligned}
b_{1,[66]} &= \sqrt{6} (b_1 + 2 b_2), & b_{2,[66]} &= 2 (b_3 + 2 b_4 + b_5), \\
d_{1,[66]} &= \sqrt{6} (b_1 + 2 b_2), & d_{2,[66]} &= 2 (b_3 + 2 b_4 + \frac{5}{6} b_5), \\
b_{1,[\bar{3}3]} &= \sqrt{6} b_1, & b_{2,[\bar{3}3]} &= 2 b_3, & b_{1,[\bar{3}6]} &= -\sqrt{3} b_5.
\end{aligned} \tag{53}$$

The result (53) implies 2 sum rules, which we already derived in (33) based on the heavy-quark spin symmetry. The third sum rule in (33) is recovered with and only with $b_5 = 0$. Again this is consistent with the scaling ansatz (43). Our results (53) are related to the previous analysis of Jenkins [21], that considered charmed baryon masses rather than matrix elements of scalar currents in the $1/N_c$ expansion. Note that the counter terms in (53) contribute to the baryon self energies at tree-level already. Therefore a comparison with the operator analysis suggested in [21] is possible. Our analysis differs from the one in [21] to the extent that our expansion is not relying on an additional expansion in a quark-mass difference.

We continue with a derivation of large- N_c sum rules for the chiral-symmetry breaking low-energy constants introduced in (17). They contribute to the time-ordered product of

two scalar currents as evaluated in the baryon states (19). At NLO in the $1/N_c$ expansion we find the relevance of 5 operators

$$\begin{aligned} \langle \bar{p}, mn_{\pm}, \bar{S}, \bar{\chi} | S^{ab}(q) | p, kl_{\pm}, S, \chi \rangle &= (mn_{\pm}, \bar{S}, \bar{\chi} | \mathcal{O}^{ab} | kl_{\pm}, S, \chi), \\ \mathcal{O}^{ab} &= (c_1 \delta_{a0} \delta_{b0} + c_2 \delta_{ab}) \mathbb{1} + c_3 (T^a \delta_{b0} + \delta_{a0} T^b) \\ &\quad + c_4 d_{abc} T^c + c_5 \{T^a, T^b\}, \end{aligned} \quad (54)$$

where at this order no spin-symmetry breaking operator has to be considered. The sum in (54) starts at $c = 1, \dots, 8$. The matrix elements of the operators in (54) are given in the Appendix. A comparison with the tree-level expressions (19) implies the matching conditions

$$\begin{aligned} c_{1,[\bar{3}\bar{3}]} &= 2c_2 + \frac{2}{3}c_5, & c_{2,[\bar{3}\bar{3}]} &= \frac{2}{3}c_1, & c_{3,[\bar{3}\bar{3}]} &= 2\sqrt{\frac{2}{3}}c_3, \\ c_{4,[\bar{3}\bar{3}]} &= c_4 + c_5, & c_{1,[\bar{3}6]} &= c_{2,[\bar{3}6]} = c_{3,[\bar{3}6]} = 0, \\ c_{1,[66]} &= e_{1,[66]} = 2c_2 + \frac{2}{3}c_5, & c_{2,[66]} &= e_{2,[66]} = \frac{2}{3}c_1, \\ c_{3,[66]} &= e_{3,[66]} = 2\sqrt{\frac{2}{3}}c_3, & c_{4,[66]} &= e_{4,[66]} = c_4 + c_5, \\ c_{5,[66]} &= e_{5,[66]} = 2c_5 - 6c_2 - 2c_4, \end{aligned} \quad (55)$$

From (55) one can deduce $17 - 5 = 12$ sum-rules. This is to be compared with the $17 - 9 = 8$ sum rules in (34) that follow from the heavy-quark spin symmetry only. Here large- N_c QCD shows its predictive power in generating 4 additional sum rules. We recover the 8 sum rules collected in (34) together with the four additional sum rules

$$c_{n,[\bar{3}\bar{3}]} = c_{n,[66]} \quad \text{for } n = 1, \dots, 4. \quad (56)$$

We close this section with a study of the time-ordered product of two axial-vector currents. The leading order operator expansion was already worked out in [18]. Here we need to consider matrix elements in charmed baryons and derive the implication for the chiral two-body interactions introduced (14). According to [18] there are 7 distinct operators

$$\begin{aligned} \langle \bar{p}, mn_{\pm}, \bar{S}, \bar{\chi} | A_{ij}^{ab}(q) | p, kl_{\pm}, S, \chi \rangle &= (mn_{\pm}, \bar{S}, \bar{\chi} | \mathcal{O}_{ij}^{ab} | kl_{\pm}, S, \chi), \\ \mathcal{O}_{ij}^{ab} &= -\delta_{ij} \left(g_1 \left(\frac{1}{3} \delta_{ab} \mathbb{1} + d_{abc} T^c \right) + \frac{1}{2} g_2 \{T^a, T^b\} \right) \\ &\quad + \frac{(\bar{p} + p)_i (\bar{p} + p)_j}{4M} \left(g_3 \left(\frac{1}{3} \delta_{ab} \mathbb{1} + d_{abc} T^c \right) + \frac{1}{2} g_4 \{T^a, T^b\} \right) \\ &\quad + \epsilon_{ijk} f_{abc} g_5 G_k^c + \frac{1}{2} g_6 \{G_i^a, G_j^b\} + \frac{1}{2} g_7 \{G_j^a, G_i^b\} + \dots, \end{aligned} \quad (57)$$

where we focus on the space components of the correlation function. In (57) we have $q = \bar{p} - p$ and $a, b = 1, \dots, 8$. In addition, we consider terms only that arise in the small-momentum expansion and that are required for the desired matching with (16). The dots in (57) represent additional terms that are further suppressed in $1/N_c$ or for small 3-momenta p and \bar{p} . Here we also assumed a degenerate baryon mass M for the baryon states as they arise in the large- N_c limit. An application of the results of our Appendix leads to the matching result already included in Tab. I, where they are compared with the consequences of the spin-symmetric interactions introduced in (35). From the operator analysis (57), we obtain 29 sum rules

$$\begin{aligned}
g_{D,[66]}^{(S)} &= \frac{3}{2} g_{0,[66]}^{(S)}, & g_{0,[\bar{3}\bar{3}]}^{(S)} &= g_{0,[66]}^{(S)}, & g_{D,[\bar{3}\bar{3}]}^{(S)} &= \frac{3}{2} g_{0,[66]}^{(S)}, & g_{1,[\bar{3}6]}^{(S)} &= g_{D,[\bar{3}6]}^{(S)} = 0, \\
h_{0,[66]}^{(S)} &= -g_{0,[66]}^{(S)} - \frac{1}{3} h_{1,[66]}^{(S)}, & h_{2,[66]}^{(S)} &= -\frac{3}{2} g_{0,[66]}^{(S)} - \frac{1}{2} h_{1,[66]}^{(S)}, \\
h_{3,[66]}^{(S)} &= \frac{3}{2} h_{1,[66]}^{(S)}, & h_{4,[66]}^{(S)} &= -g_{1,[66]}^{(S)} - h_{1,[66]}^{(S)}, & h_{5,[66]}^{(S)} &= \frac{3}{2} h_{1,[66]}^{(S)}, \\
g_{D,[66]}^{(V)} &= \frac{3}{2} g_{0,[66]}^{(V)}, & g_{0,[\bar{3}\bar{3}]}^{(V)} &= g_{0,[66]}^{(V)}, & g_{1,[\bar{3}\bar{3}]}^{(V)} &= g_{1,[66]}^{(V)}, & g_{D,[\bar{3}\bar{3}]}^{(V)} &= \frac{3}{2} g_{0,[66]}^{(V)}, \\
g_{1,[\bar{3}6]}^{(V)} &= g_{D,[\bar{3}6]}^{(V)} = 0, & h_{0,[66]}^{(V)} &= -g_{0,[66]}^{(V)}, & h_{1,[66]}^{(V)} &= -g_{1,[66]}^{(V)}, & h_{2,[66]}^{(V)} &= -\frac{3}{2} g_{0,[66]}^{(V)}, \\
f_{0,[66]}^{(A)} &= \frac{1}{\sqrt{3}} h_{1,[66]}^{(S)}, & f_{D,[66]}^{(A)} &= \frac{\sqrt{3}}{2} h_{1,[66]}^{(S)}, & f_{1,[66]}^{(A)} &= \frac{\sqrt{3}}{2} h_{1,[66]}^{(S)}, & f_{1,[\bar{3}6]}^{(A)} &= f_{D,[\bar{3}6]}^{(A)} = 0, \\
h_{F,[66]}^{(T)} &= -\frac{3}{2} g_{F,[66]}^{(T)}, & g_{F,[\bar{3}6]}^{(T)} &= -\frac{1}{2\sqrt{3}} f_{F,[\bar{3}6]}^{(A)}, & f_{F,[66]}^{(A)} &= \frac{1}{\sqrt{3}} f_{F,[\bar{3}6]}^{(A)}, & g_{F,[\bar{3}\bar{3}]}^{(T)} &= 0, \\
g_{1,[\bar{3}6]}^{(T)} &= \frac{1}{\sqrt{3}} f_{F,[\bar{3}6]}^{(A)}.
\end{aligned} \tag{58}$$

It is interesting to compare our large- N_c sum rules (58) with the sum rules implied by the heavy-quark spin symmetry in (37). The heavy-quark spin operator J_Q^i , which would violate the spin-symmetry explicitly, contributes at subleading orders in the operator expansion of (57) only. Therefore one may expect that the spin-symmetry sum rules in (37) do not provide additional constraints on the parameters. However, this is not the case. The combination of (58) with the spin-symmetry sum rules (37) does lead to 4 extra relations, that are implied by

$$g_3 = \frac{1}{2} g_4, \quad g_5 = 0, \quad g_6 = g_7 = -3 g_4. \tag{59}$$

This does not contradict the systematics of the large- N_c operator expansion. It merely shows that sometimes coefficients like $g_3 - \frac{1}{2} g_4, g_5, g_6 - g_7, g_6 + \frac{3}{4} g_4$ are suppressed in the

heavy-quark mass expansion. Though the operator analysis is not predicting such a feature, it can not exclude it.

V. SUMMARY

We constructed the chiral $SU(3)$ Lagrangian with charmed baryons with spin $J^P = 1/2^+$ and $J^P = 3/2^+$ at subleading orders as it is required for a chiral extrapolation of the charmed baryon masses from lattice QCD simulations. All counter terms that are relevant at next-to-next-to-next-to-leading order (N^3LO) for such a chiral extrapolation were identified.

- At N^2LO we find 16 low-energy parameters. There are 3 mass parameters for the anti-triplet and the two sextet baryons, 6 parameters describing the meson-baryon vertices and 7 symmetry breaking parameters. The heavy-quark spin symmetry predicts four sum rules for the meson-baryon vertices and degenerate masses for the two baryon sextet fields. Here a large- N_c operator analysis at NLO suggests the relevance of one further spin-symmetry breaking parameter.
- At N^3LO there are additional 17 chiral symmetry breaking parameters and 24 symmetry preserving parameters. For the leading symmetry conserving two-body counter terms involving two baryon fields and two Goldstone boson fields we find 36 terms. While the heavy-quark spin symmetry leads to $36 - 16 = 20$ sum rules, an expansion in $1/N_c$ at next-to-leading order (NLO) generates $36 - 7 = 29$ parameter relations. A combined expansion leaves 3 unknown parameters only, a parameter reduction by about a factor of 10. For the symmetry breaking counter terms we find 17 terms, for which there are $17 - 9 = 8$ sum rules from the heavy-quark spin symmetry and $17 - 5 = 12$ sum rules from a $1/N_c$ expansion at NLO. Here a combined expansion does not further reduce the number of parameters.

At present such sum rules can not be confronted directly with empirical information. They are useful constraints in establishing a systematic coupled-channel effective field theory for the Goldstone-boson charmed-baryon scattering beyond the threshold region. In the near future a significant increase of the lattice data base on charmed baryon masses is expected. They will allow a significant test of our results.

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VI. APPENDIX

We consider matrix elements of the symmetric product of two one-body operators \mathcal{O} in the charmed baryon ground state at $N_c = 3$. The generic notation

$$\begin{aligned}\langle \mathcal{O} \rangle_{\bar{S}S}^{\pm\pm} &\equiv (mn_{\pm}, \bar{S}, \bar{\chi} | \mathcal{O} | kl_{\pm}, S, \chi), \\ \langle \mathcal{O} \rangle_{\bar{S}S}^{\pm\mp} &\equiv (mn_{\pm}, \bar{S}, \bar{\chi} | \mathcal{O} | kl_{\mp}, S, \chi),\end{aligned}\quad (60)$$

will be applied. The results are expressed in terms of the flavour structures $\Lambda_{(kl)_+}^{(a), (rs)\pm}$ and $\Lambda_{(kl)_-}^{(a), (rs)\pm}$ and the spin structures σ_i and S_i introduced in (10). The following identities turn useful

$$\begin{aligned}\delta_{(ij)_{\pm}}^{(mn)\pm} &= \frac{1}{2} \left(\delta_{mi} \delta_{nj} \pm \delta_{ni} \delta_{mj} \right), \\ \Lambda_{(kl)_+}^{(a), (rs)\pm} \delta_{(rs)_{\pm}}^{(mn)\pm} &= \Lambda_{(kl)_+}^{(a), (mn)\pm}, \quad \Lambda_{(kl)_-}^{(a), (rs)\pm} \delta_{(rs)_{\pm}}^{(mn)\pm} = \Lambda_{(kl)_-}^{(a), (mn)\pm}.\end{aligned}\quad (61)$$

We find

$$\begin{aligned}\langle \{ J_i, J_j \} \rangle_{\frac{1}{2}\frac{1}{2}}^{++} &= \frac{4}{3} \delta_{ij} \delta_{\bar{\chi}\chi} \delta_{(kl)_+}^{(mn)+}, \quad \langle \{ J_i, T^a \} \rangle_{\frac{1}{2}\frac{1}{2}}^{++} = \frac{4}{3} \sigma_{\bar{\chi}\chi}^{(i)} \Lambda_{(kl)_+}^{(a), (mn)+}, \\ \langle \{ J_i, G_j^a \} \rangle_{\frac{1}{2}\frac{1}{2}}^{++} &= \frac{2}{3} \delta_{ij} \delta_{\bar{\chi}\chi} \Lambda_{(kl)_+}^{(a), (mn)+}, \\ \langle \{ T^a, T^b \} \rangle_{\frac{1}{2}\frac{1}{2}}^{++} &= \left(\Lambda_{(rs)_+}^{(a), (mn)+} \Lambda_{(kl)_+}^{(b), (tu)+} + \Lambda_{(rs)_+}^{(b), (mn)+} \Lambda_{(kl)_+}^{(a), (tu)+} \right) \delta_{(tu)_+}^{(rs)+} \delta_{\bar{\chi}\chi}, \\ \langle \{ T^a, G_i^b \} \rangle_{\frac{1}{2}\frac{1}{2}}^{++} &= \frac{1}{3} \sigma_{\bar{\chi}\chi}^{(i)} \left(\Lambda_{(rs)_+}^{(a), (mn)+} \Lambda_{(kl)_+}^{(b), (rs)+} + \Lambda_{(rs)_+}^{(b), (mn)+} \Lambda_{(kl)_+}^{(a), (rs)+} \right), \\ \langle \{ G_i^a, G_j^b \} \rangle_{\frac{1}{2}\frac{1}{2}}^{++} &= \frac{1}{6} \delta_{ij} \delta_{\bar{\chi}\chi} \left(\Lambda_{(rs)_+}^{(a), (mn)+} \Lambda_{(kl)_+}^{(b), (rs)+} + (a \leftrightarrow b) \right) \\ &\quad + \frac{1}{12} i \epsilon_{ijk} \sigma_{\bar{\chi}\chi}^{(k)} \left(\Lambda_{(rs)_+}^{(a), (mn)+} \Lambda_{(kl)_+}^{(b), (rs)+} - (a \leftrightarrow b) \right) \\ &\quad + \frac{1}{12} \delta_{ij} \delta_{\bar{\chi}\chi} \left(\Lambda_{(rs)_-}^{(a), (mn)+} \Lambda_{(kl)_+}^{(b), (rs)-} + (a \leftrightarrow b) \right) \\ &\quad + \frac{1}{12} i \epsilon_{ijk} \sigma_{\bar{\chi}\chi}^{(k)} \left(\Lambda_{(rs)_-}^{(a), (mn)+} \Lambda_{(kl)_+}^{(b), (rs)-} - (a \leftrightarrow b) \right), \\ \langle \{ J_Q^i, J_j \} \rangle_{\frac{1}{2}\frac{1}{2}}^{++} &= \frac{2}{3} \delta_{ij} \delta_{\bar{\chi}\chi} \delta_{(kl)_+}^{(mn)+}, \quad \langle \{ J_Q^i, J_Q^j \} \rangle_{\frac{1}{2}\frac{1}{2}}^{++} = \frac{1}{2} \delta_{ij} \delta_{\bar{\chi}\chi} \delta_{(kl)_+}^{(mn)+}, \\ \langle \{ J_Q^i, T^a \} \rangle_{\frac{1}{2}\frac{1}{2}}^{++} &= \sigma_{\bar{\chi}\chi}^{(i)} \Lambda_{(kl)_+}^{(a), (mn)+}, \quad \langle \{ J_Q^i, G_j^a \} \rangle_{\frac{1}{2}\frac{1}{2}}^{++} = \frac{1}{3} \delta_{ij} \delta_{\bar{\chi}\chi} \Lambda_{(kl)_+}^{(a), (mn)+}, \\ \langle \{ J_i, J_j \} \rangle_{\frac{3}{2}\frac{3}{2}}^{++} &= \frac{4}{3} \delta_{ij} \delta_{\bar{\chi}\chi} \delta_{(kl)_+}^{(mn)+} - \left(S_i S_j^\dagger + S_j S_i^\dagger - \frac{2}{3} \delta_{ij} \right)_{\bar{\chi}\chi} \delta_{(kl)_+}^{(mn)+}, \\ \langle \{ J_i, T^a \} \rangle_{\frac{3}{2}\frac{3}{2}}^{++} &= 2 (\vec{S} \sigma_i \vec{S}^\dagger)_{\bar{\chi}\chi} \Lambda_{(kl)_+}^{(a), (mn)+}, \\ \langle \{ J_i, G_j^a \} \rangle_{\frac{3}{2}\frac{3}{2}}^{++} &= \frac{2}{3} \delta_{ij} \delta_{\bar{\chi}\chi} \Lambda_{(kl)_+}^{(a), (mn)+} - \frac{1}{2} \left(S_i S_j^\dagger + S_j S_i^\dagger - \frac{2}{3} \delta_{ij} \right)_{\bar{\chi}\chi} \Lambda_{(kl)_+}^{(a), (mn)+},\end{aligned}$$

$$\begin{aligned}
\langle \{ T^a, T^b \} \rangle_{\frac{3}{2}\frac{3}{2}}^{++} &= \left(\Lambda_{(rs)_+}^{(a), (mn)_+} \Lambda_{(kl)_+}^{(b), (tu)_+} + \Lambda_{(rs)_+}^{(b), (mn)_+} \Lambda_{(kl)_+}^{(a), (tu)_+} \right) \delta_{(tu)_+}^{(rs)_+} \delta_{\bar{\chi}\chi}, \\
\langle \{ T^a, G_i^b \} \rangle_{\frac{3}{2}\frac{3}{2}}^{++} &= \frac{1}{2} (\vec{S} \sigma_i \vec{S}^\dagger)_{\bar{\chi}\chi} \left(\Lambda_{(rs)_+}^{(a), (mn)_+} \Lambda_{(kl)_+}^{(b), (rs)_+} + \Lambda_{(rs)_+}^{(b), (mn)_+} \Lambda_{(kl)_+}^{(a), (rs)_+} \right), \\
\langle \{ G_i^a, G_j^b \} \rangle_{\frac{3}{2}\frac{3}{2}}^{++} &= \frac{1}{6} \delta_{ij} \delta_{\bar{\chi}\chi} \left(\Lambda_{(rs)_+}^{(a), (mn)_+} \Lambda_{(kl)_+}^{(b), (rs)_+} + (a \leftrightarrow b) \right) \\
&\quad + \frac{1}{8} i \epsilon_{ijk} (\vec{S} \sigma^{(k)} \vec{S}^\dagger)_{\bar{\chi}\chi} \left(\Lambda_{(rs)_+}^{(a), (mn)_+} \Lambda_{(kl)_+}^{(b), (rs)_+} - (a \leftrightarrow b) \right) \\
&\quad - \frac{1}{8} (S_i S_j^\dagger + S_j S_i^\dagger - \frac{2}{3} \delta_{ij})_{\bar{\chi}\chi} \left(\Lambda_{(rs)_+}^{(a), (mn)_+} \Lambda_{(kl)_+}^{(b), (rs)_+} + (a \leftrightarrow b) \right) \\
&\quad + \frac{1}{12} \delta_{ij} \delta_{\bar{\chi}\chi} \left(\Lambda_{(rs)_-}^{(a), (mn)_+} \Lambda_{(kl)_+}^{(b), (rs)_-} + (a \leftrightarrow b) \right) \\
&\quad + \frac{1}{8} i \epsilon_{ijk} (\vec{S} \sigma^{(k)} \vec{S}^\dagger)_{\bar{\chi}\chi} \left(\Lambda_{(rs)_-}^{(a), (mn)_+} \Lambda_{(kl)_+}^{(b), (rs)_-} - (a \leftrightarrow b) \right) \\
&\quad + \frac{1}{8} (S_i S_j^\dagger + S_j S_i^\dagger - \frac{2}{3} \delta_{ij})_{\bar{\chi}\chi} \left(\Lambda_{(rs)_-}^{(a), (mn)_+} \Lambda_{(kl)_+}^{(b), (rs)_-} + (a \leftrightarrow b) \right), \\
\langle \{ J_Q^i, J_j \} \rangle_{\frac{3}{2}\frac{3}{2}}^{++} &= \frac{5}{9} \delta_{ij} \delta_{\bar{\chi}\chi} \delta_{(kl)_+}^{(mn)_+} - \frac{2}{3} \left(S_i S_j^\dagger + S_j S_i^\dagger - \frac{2}{3} \delta_{ij} \right)_{\bar{\chi}\chi} \delta_{(kl)_+}^{(mn)_+}, \\
\langle \{ J_Q^i, J_Q^j \} \rangle_{\frac{3}{2}\frac{3}{2}}^{++} &= \frac{5}{18} \delta_{ij} \delta_{\bar{\chi}\chi} \delta_{(kl)_+}^{(mn)_+} - \frac{1}{3} \left(S_i S_j^\dagger + S_j S_i^\dagger - \frac{2}{3} \delta_{ij} \right)_{\bar{\chi}\chi} \delta_{(kl)_+}^{(mn)_+}, \\
\langle \{ J_Q^i, T^a \} \rangle_{\frac{3}{2}\frac{3}{2}}^{++} &= (\vec{S} \sigma^{(i)} \vec{S}^\dagger)_{\bar{\chi}\chi} \Lambda_{(kl)_+}^{(a), (mn)_+}, \\
\langle \{ J_Q^i, G_j^a \} \rangle_{\frac{3}{2}\frac{3}{2}}^{++} &= \frac{5}{18} \delta_{ij} \delta_{\bar{\chi}\chi} \Lambda_{(kl)_+}^{(a), (mn)_+} - \frac{1}{3} \left(S_i S_j^\dagger + S_j S_i^\dagger - \frac{2}{3} \delta_{ij} \right)_{\bar{\chi}\chi} \Lambda_{(kl)_+}^{(a), (mn)_+},
\end{aligned}$$

$$\begin{aligned}
\langle \{ J_i, J_j \} \rangle_{\frac{1}{2}\frac{1}{2}}^{--} &= 0, \quad \langle \{ J_i, T^a \} \rangle_{\frac{1}{2}\frac{1}{2}}^{--} = 0, \quad \langle \{ J_i, G_j^b \} \rangle_{\frac{1}{2}\frac{1}{2}}^{--} = 0, \\
\langle \{ T^a, T^b \} \rangle_{\frac{1}{2}\frac{1}{2}}^{--} &= \left(\Lambda_{(rs)_+}^{(a), (mn)_-} \Lambda_{(kl)_-}^{(b), (tu)_+} + \Lambda_{(rs)_+}^{(b), (mn)_-} \Lambda_{(kl)_-}^{(a), (tu)_+} \right) \delta_{(tu)_-}^{(rs)_-} \delta_{\bar{\chi}\chi}, \\
\langle \{ T^a, G_i^b \} \rangle_{\frac{1}{2}\frac{1}{2}}^{--} &= 0, \quad \langle \{ J_Q^i, J_j \} \rangle_{\frac{1}{2}\frac{1}{2}}^{--} = 0, \quad \langle \{ J_Q^i, G_j^a \} \rangle_{\frac{1}{2}\frac{1}{2}}^{--} = 0, \\
\langle \{ G_i^a, G_j^b \} \rangle_{\frac{1}{2}\frac{1}{2}}^{--} &= \frac{1}{4} \delta_{ij} \delta_{\bar{\chi}\chi} \left(\Lambda_{(rs)_+}^{(a), (mn)_-} \Lambda_{(kl)_-}^{(b), (rs)_+} + \Lambda_{(rs)_+}^{(b), (mn)_-} \Lambda_{(kl)_-}^{(a), (rs)_+} \right), \\
\langle \{ J_Q^i, J_Q^j \} \rangle_{\frac{1}{2}\frac{1}{2}}^{--} &= \frac{1}{2} \delta_{ij} \delta_{\bar{\chi}\chi} \delta_{(kl)_-}^{(mn)_-}, \quad \langle \{ J_Q^i, T^a \} \rangle_{\frac{1}{2}\frac{1}{2}}^{--} = \sigma_{\bar{\chi}\chi}^{(i)} \Lambda_{(kl)_-}^{(a), (mn)_-},
\end{aligned}$$

$$\begin{aligned}
\langle \{ J_i, J_j \} \rangle_{\frac{3}{2}\frac{1}{2}}^{++} &= -\frac{1}{\sqrt{3}} \left(S_i \sigma_j + S_j \sigma_i \right)_{\bar{\chi}\chi} \delta_{(kl)_+}^{(mn)_+}, \\
\langle \{ J_i, T^a \} \rangle_{\frac{3}{2}\frac{1}{2}}^{++} &= -\frac{2}{\sqrt{3}} S_{\bar{\chi}\chi}^{(i)} \Lambda_{(kl)_+}^{(a), (mn)_+}, \\
\langle \{ J_i, G_j^a \} \rangle_{\frac{3}{2}\frac{1}{2}}^{++} &= -\frac{1}{2\sqrt{3}} \left(S_i \sigma_j + S_j \sigma_i \right)_{\bar{\chi}\chi} \Lambda_{(kl)_+}^{(a), (mn)_+}, \\
\langle \{ T^a, G_i^b \} \rangle_{\frac{3}{2}\frac{1}{2}}^{++} &= -\frac{1}{2\sqrt{3}} S_{\bar{\chi}\chi}^{(i)} \left(\Lambda_{(rs)_+}^{(a), (mn)_+} \Lambda_{(kl)_+}^{(b), (rs)_+} + \Lambda_{(rs)_+}^{(b), (mn)_+} \Lambda_{(kl)_+}^{(a), (rs)_+} \right), \\
\langle \{ G_i^a, G_j^b \} \rangle_{\frac{3}{2}\frac{1}{2}}^{++} &= -\frac{1}{8\sqrt{3}} i \epsilon_{ijk} S_{\bar{\chi}\chi}^{(k)} \left(\Lambda_{(rs)_+}^{(a), (mn)_+} \Lambda_{(kl)_+}^{(b), (rs)_+} - (a \leftrightarrow b) \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{8\sqrt{3}} \left(S_i \sigma_j + S_j \sigma_i \right)_{\bar{\chi}\chi} \left(\Lambda_{(rs)+}^{(a),(mn)+} \Lambda_{(kl)+}^{(b),(rs)+} + (a \leftrightarrow b) \right) \\
& -\frac{1}{8\sqrt{3}} i \epsilon_{ijk} S_{\bar{\chi}\chi}^{(k)} \left(\Lambda_{(rs)-}^{(a),(mn)+} \Lambda_{(kl)+}^{(b),(rs)-} - (a \leftrightarrow b) \right) \\
& +\frac{1}{8\sqrt{3}} \left(S_i \sigma_j + S_j \sigma_i \right)_{\bar{\chi}\chi} \left(\Lambda_{(rs)-}^{(a),(mn)+} \Lambda_{(kl)+}^{(b),(rs)-} + (a \leftrightarrow b) \right), \\
\langle \{ J_Q^i, J_j \} \rangle_{\frac{3}{2}\frac{1}{2}}^{++} &= -\frac{2}{3\sqrt{3}} \left(i \epsilon_{ijk} S_{\bar{\chi}\chi}^{(k)} + \frac{1}{2} (S_i \sigma_j + S_j \sigma_i)_{\bar{\chi}\chi} \right) \delta_{(kl)+}^{(mn)+}, \\
\langle \{ J_Q^i, G_j^a \} \rangle_{\frac{3}{2}\frac{1}{2}}^{++} &= -\frac{1}{3\sqrt{3}} \left(i \epsilon_{ijk} S_{\bar{\chi}\chi}^{(k)} + \frac{1}{2} (S_i \sigma_j + S_j \sigma_i)_{\bar{\chi}\chi} \right) \Lambda_{(kl)+}^{(a),(mn)+}, \\
\langle \{ J_i, J_j \} \rangle_{\frac{3}{2}\frac{1}{2}}^{+-} &= 0, \quad \langle \{ J_i, T^a \} \rangle_{\frac{3}{2}\frac{1}{2}}^{+-} = 0, \quad \langle \{ J_Q^i, J_j \} \rangle_{\frac{3}{2}\frac{1}{2}}^{+-} = 0, \\
\langle \{ J_i, G_j^a \} \rangle_{\frac{3}{2}\frac{1}{2}}^{+-} &= -\frac{1}{2} i \epsilon_{ijk} S_{\bar{\chi}\chi}^{(k)} \Lambda_{(kl)-}^{(a),(mn)+}, \\
\langle \{ T^a, G_i^b \} \rangle_{\frac{3}{2}\frac{1}{2}}^{+-} &= -\frac{1}{2} S_{\bar{\chi}\chi}^{(i)} \left(\Lambda_{(rs)+}^{(a),(mn)+} \Lambda_{(kl)-}^{(b),(rs)+} + \Lambda_{(rs)-}^{(b),(mn)+} \Lambda_{(kl)-}^{(a),(rs)-} \right), \\
\langle \{ G_i^a, G_j^b \} \rangle_{\frac{3}{2}\frac{1}{2}}^{+-} &= -\frac{1}{4} i \epsilon_{ijk} S_{\bar{\chi}\chi}^{(k)} \left(\Lambda_{(rs)+}^{(a),(mn)+} \Lambda_{(kl)-}^{(b),(rs)+} - \Lambda_{(rs)+}^{(b),(mn)+} \Lambda_{(kl)-}^{(a),(rs)+} \right), \\
\langle \{ J_Q^i, G_j^a \} \rangle_{\frac{3}{2}\frac{1}{2}}^{+-} &= -\frac{1}{3} \left(i \epsilon_{ijk} S_{\bar{\chi}\chi}^{(k)} + \frac{1}{2} (S_i \sigma_j + S_j \sigma_i)_{\bar{\chi}\chi} \right) \Lambda_{(kl)-}^{(a),(mn)+}, \\
\langle \{ J_i, J_j \} \rangle_{\frac{1}{2}\frac{1}{2}}^{-+} &= 0, \quad \langle \{ J_i, T^a \} \rangle_{\frac{1}{2}\frac{1}{2}}^{-+} = 0, \quad \langle \{ J_Q^i, J_j \} \rangle_{\frac{1}{2}\frac{1}{2}}^{-+} = 0, \\
\langle \{ J_i, G_j^a \} \rangle_{\frac{1}{2}\frac{1}{2}}^{-+} &= \frac{1}{2\sqrt{3}} i \epsilon_{ijk} \sigma_{\bar{\chi}\chi}^{(k)} \Lambda_{(kl)+}^{(a),(mn)-}, \\
\langle \{ T^a, G_i^b \} \rangle_{\frac{1}{2}\frac{1}{2}}^{-+} &= -\frac{1}{2\sqrt{3}} \sigma_{\bar{\chi}\chi}^{(i)} \left(\Lambda_{(rs)-}^{(a),(mn)-} \Lambda_{(kl)+}^{(b),(rs)-} + \Lambda_{(rs)+}^{(b),(mn)-} \Lambda_{(kl)+}^{(a),(rs)+} \right), \\
\langle \{ G_i^a, G_j^b \} \rangle_{\frac{1}{2}\frac{1}{2}}^{-+} &= -\frac{1}{4\sqrt{3}} i \epsilon_{ijk} \sigma_{\bar{\chi}\chi}^{(k)} \left(\Lambda_{(rs)+}^{(a),(mn)-} \Lambda_{(kl)+}^{(b),(rs)+} - \Lambda_{(rs)+}^{(b),(mn)-} \Lambda_{(kl)+}^{(a),(rs)+} \right), \\
\langle \{ J_Q^i, G_j^a \} \rangle_{\frac{1}{2}\frac{1}{2}}^{-+} &= -\frac{1}{2\sqrt{3}} \delta_{ij} \delta_{\bar{\chi}\chi} \Lambda_{(kl)+}^{(a),(mn)-}. \tag{62}
\end{aligned}$$

Note that all results of this Appendix are valid also for the singlet components with $a = 0$ or $b = 0$ if the convention (41) is used together with $\lambda_0 = \sqrt{2/3} \mathbf{1}$ in (10).

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